32. Mon, Apr. 7

Last time, we introduced the genus g surfaces M_g, defined as the g-fold connected sum of copies of T^2. We found that

$$\pi_1(M_g) \cong F(a_1, b_1, \dots, a_g, b_g)/[a_1, b_1] \dots [a_g, b_g].$$

Proposition 32.1. $\pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$.

Proof. Let $F = F(a_1, b_1, \dots, a_n, b_n)$, $G = \pi_1(M_g)$, and let $q : F \longrightarrow G$ be the quotient. Then $[a_1, b_1] \dots [a_g, b_g] \in [F, F]$ and $q([F, F]) = [G, G]$, so the Third Isomorphism Theorem gives $G_{ab} \cong F_{ab} \cong \mathbb{Z}^{2g}$. ■

Corollary 32.2. *If $g_1 \neq g_2$ them $M_{g_1} \not\cong M_{g_2}$.*

We have distinguished the fundamental groups $\pi_1(M_g)$ from each other and also from $\pi_1(\mathbb{RP}^2)$.

What about the Klein bottle K? We found last week that $\pi_1(K) \cong F(a, b)/aba^{-1}b$. If we abelianize this fundamental group, we get

$$\mathbb{Z}[a] \times \mathbb{Z}[b]/(a + b - a + b) = \mathbb{Z}[a] \times \mathbb{Z}[b]/2b \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

This group is different from all of the others, so K is not homeomorphic to any of the above surfaces. The last main example is

Example 32.3. ($\mathbb{RP}^2 \# \mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$) Suppose we take a connect sum of g copies of $\mathbb{RP}^2$. We will call this surface N_g. Following the examples from last week, we see that we get a fundamental group of

$$\pi_1(N_g) \cong F(a_1, \dots, a_g)/a_1^2 a_2^2 \dots a_g^2$$

and $\chi(N_g) = 1 - g + 1 = 2 - g$. The abelianization is then

$$\pi_1(N_g)_{ab} \cong \mathbb{Z}^g/(2, 2, \dots, 2).$$

Define a homomorphism $\varphi : \mathbb{Z}^g/(2, \dots, 2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{g-1}$ by

$$\varphi(n_1, \dots, n_g) = (n_1, n_2 - n_1, n_3 - n_1, \dots, n_g - n_1).$$

Then it is easily verified that φ is an isomorphism. In other words,

$$\pi_1(N_g)_{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}^{g-1}.$$

We also compute that

$$\chi(N_g) = 1 - g + 1 = 2 - g.$$

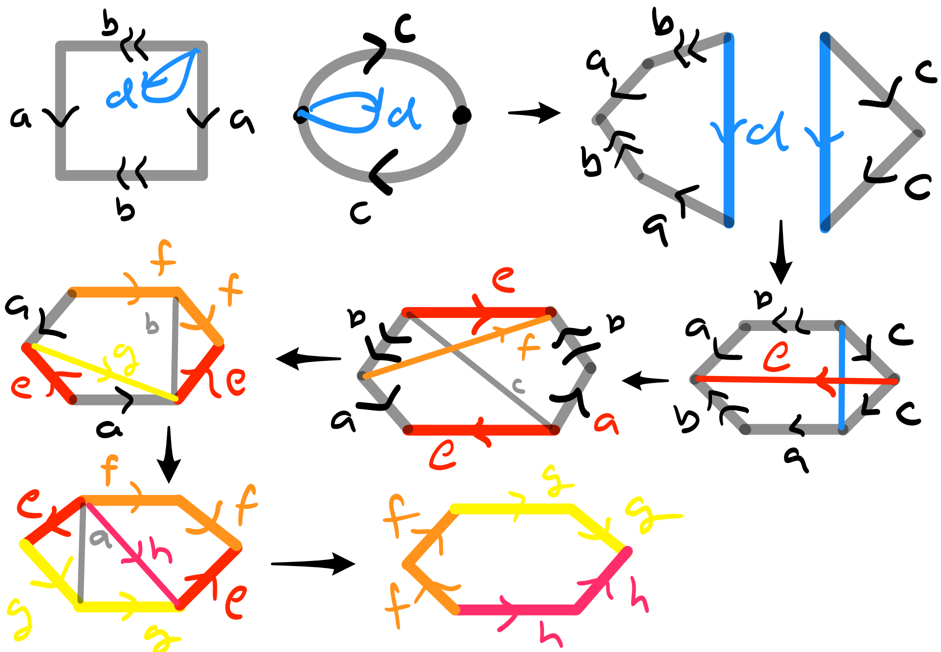
Ok, so we have argued that the compact surfaces S^2, M_g ($g \geq 1$), and N_g ($g \geq 1$) all have different fundamental groups and thus are not homeomorphic. The remarkable fact is that these are *all* of the compact (connected) surfaces.

Theorem 32.4. *Every compact, connected surface is homeomorphic to some M_g, $g \geq 0$ or to some N_g, $g \geq 1$.*

Corollary 32.5. *If $\chi(M) = n$ is odd, then $M \cong N_{2-n}$*

Lemma 32.6. $T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

The proof is in the picture:



In particular, this implies that $M_g \# N_k \cong N_{2g+k}$.

33. Wed, Apr. 9

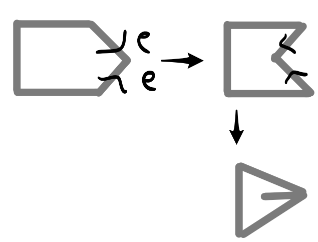
Proof of the theorem. Let M be a compact, connected surface. **We assume without proof** (see Prop 6.14 from Lee) that

- M is a 2-cell complex with a single 2-cell.
- the attaching map $\alpha : S^1 \longrightarrow M^1$ for the 2-cell has the following property: let U be the interior of a 1-cell. Then the restriction $\alpha : \alpha^{-1}(U) \longrightarrow U$ is a double cover. In other words, if we label ∂D^2 according to the edge identifications as we have done in the examples, each edge appears exactly twice. Note that this must happen since each interior point on the edge needs to have a half-disk on two sides.

So we can visualize M as a quotient of a $2n$-sided polygon.

As we said above, each edge appears exactly twice on the boundary of the two-cell. If the two occurrences have **opposite** orientations (as in the sphere), we say the pair is an **oriented pair**. If the two occurrences have the **same** orientation (as in $\mathbb{RP}^2$), we say this is a **twisted** pair. There will be 4 reductions in the proof!!

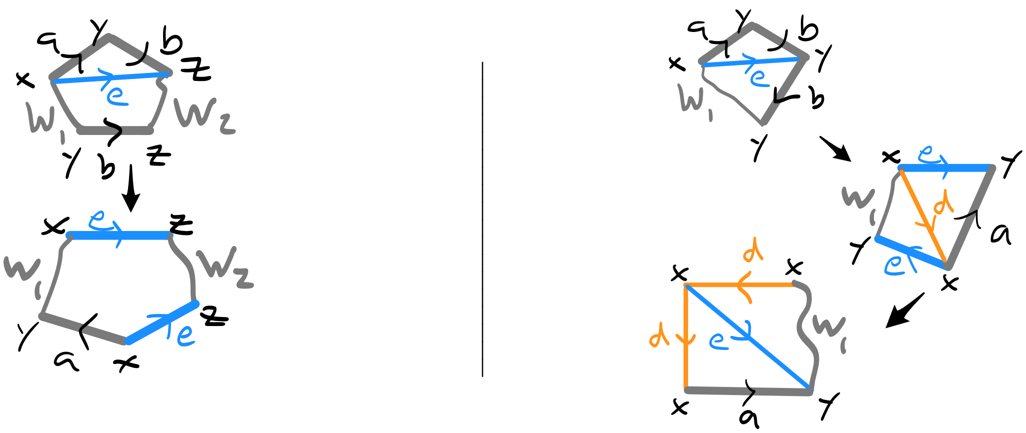
(1) If $M \cong S^2$, we are done, so suppose (for the rest of the proof) this is *not* the case. Then we can reduce to a cell structure with no *adjacent oriented pairs*. (Just fold these together.)



(2) We can reduce to a cell structure where all twisted pairs are adjacent.

If this creates any adjacent oriented pairs, fold them in.

(3) We can reduce to a cell structure with a single 0-cell. Suppose a is an edge from x to y and that $x \neq y$. Let b be the other edge connecting to y. By (1), b can't be a^{-1}. If $b = a$ then $x = y$. Suppose $b \neq a$, and write z for the other vertex on b. Then the edge b must occur somewhere else on the boundary. We use the moves in the pictures below, depending on whether the pair b is oriented or twisted.



This converts a vertex y into a vertex x. Note that this procedure does not separate any adjacent twisted pairs, since the adjacent twisted pair b gets replaced by d.

(4) Observe that any oriented pair a, a^{-1} is interlaced with another oriented pair b, b^{-1}. If not, we can write the boundary in the form $aW_1a^{-1}W_2$. Now, given our assumption and previous steps, no edge in W_1 gets identified with an edge in W_2. It follows that if the endpoints of a are x and y, then these two vertices never get identified with each other, as the vertex x cannot appear in W_1 and similarly y cannot appear in W_2.

34. Fri, Apr. 11

(5) We can further arrange it so that there is no interference: the oriented pairs of edges occur as $aba^{-1}b^{-1}$ with no other edges in between. The proof is in the picture below, taken from p. 177 of Lee.

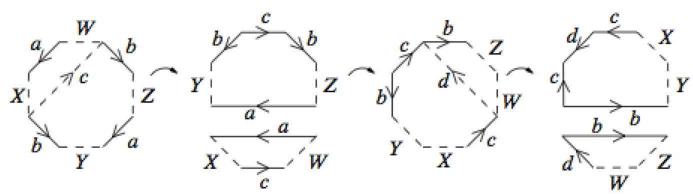


Fig. 6.22: Bringing intertwined complementary pairs together.

Now we are done by Lemma 32.6. M is homeomorphic either to a connect sum of projective planes or to a connect sum of tori. ■

We saw in Corollary 32.5 that if $\chi(M)$ is odd, we can immediately identify the homeomorphism type of M. If $\chi(M)$ is even, this is not the case, as T^2 and K both have Euler characteristic equal to 0. To handle the even case, we make a definition.

Say that a surface M is **orientable** if it has a cell structure as above with no twisted pairs of edges.

Proposition 34.1. *A surface is orientable if and only if it is homeomorphic to some M_g.*

Proof. ($\Leftarrow$) Our standard cell structures for these surfaces have no twisted pairs of edges. ($\Rightarrow$) Apply the algorithm described in the above proof, starting with only oriented pairs of edges. Step 1 does not introduce any new edges. Step 2 can be skipped. Steps 3 cuts-and-pastes along a pair of oriented edges and so does not change the orientation of any edges. Step 4 does not change the surface. Step 5 again only cuts-and-pastes along oriented edges. It follows that in reducing to standard form, we do not introduce any twisted pairs of edges. ■

In fact, you should be able to convince yourself that a surface is orientable if and only if *every* cell structure as above has not twisted pairs. The point is that if you start with a cell structure involving some twisted pairs and you perform the reductions described in the proof, you will never get rid of any twisted pairs of edges.

The fact that the M_g can be embedded in $\mathbb{R}^3$ whereas the N_g cannot is precisely related to orientability. In general, you can embed a (smooth) n-dimensional manifold in $\mathbb{R}^{2n}$, but you can improve this to $\mathbb{R}^{2n-1}$ if the manifold is orientable. The definition we have given here depends on particular kinds of CW structures, but the usual definitions of orientability (in terms of homology) apply more widely.

In addition to the N_g's, the Möbius band is a 2-manifold that is famously non-orientable.

This completes the main portion of the course, in terms of what would be covered in the prelim exam. Now we can get to the fun stuff!

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We have just been studying surfaces and have determined (well, at least given presentations for) their fundamental groups. We have also seen (on exam 2) that there are higher homotopy groups $\pi_n(X)$, so we might ask about the groups $\pi_n(M_g)$ and $\pi_n(N_k)$.

Recall, again from the exam, that any covering $E \longrightarrow B$ induces an isomorphism on all higher homotopy groups. So it suffices to understand the universal covers of these surfaces.

The first example would be $M_0 = S^2$, which is simply-connected. Note that this space is also the universal cover of $N_1 = \mathbb{RP}^2$, so these will have the same higher homotopy groups. We will come back to these on Monday.

Another example is the componentwise-exponential covering $q \times q : \mathbb{R}^2 \longrightarrow T^2$, which shows that T^2 has no higher homotopy groups. Note that also could have deduced this using that

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$$

and that S^1 has no higher homotopy groups (also from Exam 2).

What about the Klein bottle K? Well, consider the relation on T^2 given by $(x, y) \sim (x+\frac{1}{2}, 1-y)$. The quotient $T^2/\sim$ is K, and the quotient map $T^2 \longrightarrow K$ is a double cover. It follows that the universal cover of T^2, which is $\mathbb{R}^2$, is also the universal cover of K. So K also has no higher homotopy groups!

It turns out the same is true for $M_{\geq 1}$ (has universal cover $\mathbb{R}^2$) and $N_{\geq 2}$.

Proposition 34.2. *If $g \geq 1$, then there is a double cover of N_g by M_{g-1}. So the universal cover of N_g is S^2 when $g = 1$ and $\mathbb{R}^2$ when $g > 1$.*