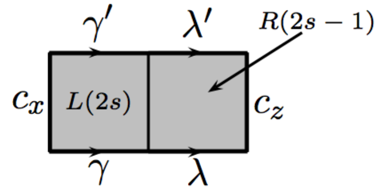
Last time, we introduced a "path-composition" operation (concatenation). The path $\gamma \cdot \lambda$ means first travel along γ in double time, then travel along λ in double time.

Proposition 3.1. *The above operation only depends on path-homotopy classes. That is, if $\gamma \simeq_p \gamma'$ and $\lambda \simeq_p \lambda'$, then $\gamma \cdot \lambda \simeq_p \gamma' \cdot \lambda'$.*

Proof. Let $L : \gamma \simeq_p \gamma'$ and $R : \lambda \simeq_p \lambda'$ be path-homotopies. We define a new path homotopy by

$$H(s,t) = \begin{cases} L(2s,t) & s \in [0,1/2] \\ R(2s-1,t) & s \in [1/2,1]. \end{cases}$$

■



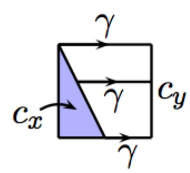
This tells us that the concatenation operation is well-defined on path-homotopy classes. We will next check that it gives a well-behaved algebraic operation. For any point $x \in X$, we denote by c_x the **constant path** at x in X.

Proposition 3.2. *Let γ (from x to y), λ, and μ be composable paths in X. Concatenation of path-homotopy classes satisfies the following properties.*

(1) *(unit law)* $[c_x] \cdot [\gamma] = [\gamma] = [\gamma] \cdot [c_y]$
(2) *(associativity)* $([\gamma] \cdot [\lambda]) \cdot [\mu] = [\gamma] \cdot ([\lambda] \cdot [\mu])$
(3) *(inverses) Define $\overline{\gamma}(s) = \gamma(1-s)$. Then $[\gamma] \cdot [\overline{\gamma}] = [c_x]$ and $[\overline{\gamma}] \cdot [\gamma] = [c_y]$.*

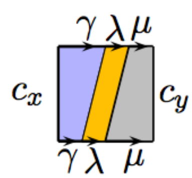
Proof. (1) Define

$$h(s,t) = \begin{cases} x & 2s \in [0,1-t] \\ \gamma(\frac{2s-1+t}{1+t}) & 2s \in [1-t,2]. \end{cases}$$



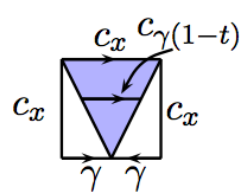
(2) Define

$$h(s,t) = \begin{cases} \gamma(\frac{4s}{1+t}) & s \in [0, \frac{1+t}{4}] \\ \lambda(4s-1-t) & s \in \left[\frac{1+t}{4}, \frac{2+t}{4}\right] \\ \mu(\frac{4s-2-t}{2-t}) & s \in [\frac{2+t}{4}, 1]. \end{cases}$$



(3) Define

$$h(s,t) = \begin{cases} \gamma(2s) & 2s \in [0,1-t] \\ \gamma(1-t) & 2s \in [1-t,1+t] \\ \gamma(2(1-s)) & 2s \in [1+t,2]. \end{cases}$$



Actually, for parts (1) and (2) there is a slicker approach, (this is in Hatcher). A **reparametrization** of a path γ is a composition $\gamma \circ \varphi$, where $\varphi : I \longrightarrow I$ is any map satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. It is clear that any such φ is homotopic to the identity map of I (just use a straight-line homotopy). For (1), we can write $c_x \cdot \gamma$ as a reparametrization of γ. Thus $c_x \cdot \gamma = \gamma \circ \varphi \simeq_p \gamma$. A similar argument also works for (2). ■

Ok, now we know that we have a group structure on $\pi_1(X, x_0)$!

So far, we have only seen examples of trivial fundamental groups. Our first major result in the course will be the computation of the fundamental group of the circle. In particular, we will show that it is nontrivial! The argument will involve a number of new ideas, and one thing I hope you will learn from this course is that **computing fundamental groups is hard**!

4. Fri, Jan. 24

Today, we will discuss the fundamental group of S^1. We will need the following technical result that could have (should have) been included in the fall semester.

Proposition 4.1. *(Lebesgue number lemma)[Munkres, 27.5 or Lee, 7.18] Let $\mathcal{U}$ be an open cover of a compact metric space X. Then there is a number $\delta > 0$ such that any subset $A \subseteq X$ of diameter less than δ is contained in an open set from the cover.*

For any n, consider the loop in S^1 given by $\gamma_n(t) = e^{2\pi int}$. For today, we will denote the standard basepoint of S^1, the point $(1,0)$, by the symbol $\star$.

Theorem 4.2. *The assignment $n \mapsto \gamma_n$ is an isomorphism of groups*

$$\Gamma : \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, \star).$$

Proof. Let's start by showing that it is a homomorphism. On your next homework assignment, you are asked to show that if G is any topological group with unit e, then composition of loops in $\pi_1(G, e)$ agrees with pointwise multiplication of loops. But S^1 is a topological group (the group of complex numbers of norm 1). This means that the path-composite $\gamma_n \cdot \gamma_k$ is path-homotopic to the loop defined by

$$\delta(t) = e^{2\pi int}e^{2\pi ikt} = e^{2\pi i(n+k)t} = \gamma_{n+k}(t).$$

To show that Γ is also a bijection, we will rely on the exponential map

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ t &\mapsto e^{2\pi it}. \end{aligned}$$

Note that $p^{-1}(\star) = \mathbb{Z}$. One important property of this map that we will need is that we can cover S^1, say using the open sets $U_1 = S^1 \setminus \{(1,0)\}$ and $U_2 = S^1 \setminus \{(-1,0)\}$. On each of these open sets U_i, the preimage $p^{-1}(U_i)$ is a countable disjoint union of subsets $V_{i,j}$ of $\mathbb{R}$, and p restricts to a homeomorphism $p : V_{i,j} \cong U_i$.

If $f : X \longrightarrow S^1$ is a map from some space X, then by a **lift** $\tilde{f} : X \longrightarrow \mathbb{R}$ we mean simply a map such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Lemma 4.3. *Let $\gamma : I \longrightarrow S^1$ be a loop at $\star$ and let $n \in \mathbb{Z}$. Then there is a unique lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ such that $\tilde{\gamma}(0) = n$.*

Proof. By the Lebesgue number lemma applied to I, there is a subdivision of I into subintervals $[s_i, s_{i+1}]$ such that each subinterval is contained in a single $\gamma^{-1}(U_i)$.

Consider the first such subinterval $[0, s_1] \subseteq \gamma^{-1}(U_2)$. Now our lifting problem simplifies to that on the right. The interval $[0, s_1]$ is connected, so the image of $\tilde{\gamma}$ must lie in a single component $V_{1,j}$. And we have no choice of the component since we have already decided that $\tilde{\gamma}(0)$ must be n. Call the component $V_{2,0}$.

$$\begin{tikzcd} & \amalg V_2, j \arrow[d, "p"] \\ {[0, s_1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & U_2 \end{tikzcd}$$

Now our lifting problem reduces to lifting against the *homeomorphism* $p_{2,0} : V_{2,0} \cong U_2$, and we define our lift on $[0, s_1]$ to be the composite $p_{2,0}^{-1} \circ \gamma$. Now play the same game with the next interval $[s_1, s_2]$. We already have a lift at the point s_1, so this forces the choice of component at this stage.

By induction, at each stage we have a unique choice of lift on the subinterval $[s_k, s_{k+1}]$. Piecing these all together gives the desired lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$. ■

Thus given a loop γ at $\star$, there is a unique lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ that starts at 0. The endpoint of the lift $\tilde{\gamma}$ must also be in $p^{-1}(0) = \mathbb{Z}$. We claim that the function $\gamma \mapsto w(\gamma) = \tilde{\gamma}(1)$ is inverse to Γ. First we must show it is well-defined.

Lemma 4.4. *Let $h : \gamma \simeq_p \delta$ be a path-homotopy between loops at $\star$ in S^1. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow \mathbb{R}$ such that $\tilde{h}(0,0) = 0$.*

Proof. We already know about the unique lift $\tilde{\gamma}$ on $I \times 0$. It is also clear that the only possible lift on $0 \times I$ is the constant lift. Now use the Lebesgue number lemma again to subdivide the compact square $I \times I$ so that every subsquare is mapped by γ into one of the U_i. Using the same argument as above, we get a unique lift on each subsquare, starting from the bottom left square and moving along each row systematically. ■

Note that the lift $\tilde{h}$ is a path-homotopy between the lifts $\tilde{\gamma}$ and $\tilde{\delta}$. This is because $\tilde{h}(0,t)$ and $\tilde{h}(1,t)$ are lifts of constant paths. By the uniqueness of lifts, according to Lemma 4.3, the lift of a constant path must be a constant path. It follows that $\tilde{\gamma}(1) = \tilde{\delta}(1)$. This shows that the function $w : \pi_1(S^1) \longrightarrow \mathbb{Z}$ is well-defined.

(To be continued on Monday…)