8. Mon, Feb. 3

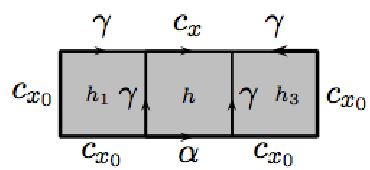
Snow Day!!

9. Wed, Feb. 5

Today, we're going to calculate $\pi_1(\mathbb{RP}^2)$, but first I want to discuss a result about contractibility of paths.

Proposition 9.1. (1) *Let $\alpha \in \pi_1(X, x_0)$. Then $\alpha \simeq_p c_{x_0}$ if and only if $\alpha : S^1 \longrightarrow X$ extends to a map $D^2 \longrightarrow X$.*
(2) *Let α and β be paths in X from x to y. Then $\alpha \simeq_p \beta$ if and only if the loop $\alpha * \overline{\beta}$ is null.*

Proof.
(1) ($\Rightarrow$) This follows from Homework II.4.
($\Leftarrow$) Again using Homework II.4, we may assume given a homotopy $h : \alpha \simeq c_x$. Since h is not assumed to be a path-homotopy, the formula $\gamma(s) = h(0, s)$ defines a possible nontrivai path. The picture



where $h_1(s,t) = \gamma(st)$ and $h_3(s,t) = \overline{\gamma}(st)$, defines a path-homotopy $H : \alpha \simeq_p \gamma \cdot c_x \cdot \overline{\gamma}$.
(2) The point is that

$$\alpha \simeq_q \beta \qquad \Rightarrow \qquad \alpha\overline{\beta} \simeq_p \beta\overline{\beta} \simeq_p c_x$$

and similarly

$$\alpha\overline{\beta} \simeq_q c_x \qquad \Rightarrow \qquad \alpha \simeq_q \alpha\overline{\beta}\beta \simeq_p c_x\beta \simeq_p \beta$$

■

Recall that for S^1, the exponential map $p : \mathbb{R} \longrightarrow S^1$ was key. The analogue of that map for $\mathbb{RP}^2$ will be the quotient map

$$q : S^2 \longrightarrow \mathbb{RP}^2.$$

Note that in this case, the "fiber" (the preimage of the basepoint) consists of two points. Another ingredient that was used for S^1 was that it has a nice cover. The same is true for $\mathbb{RP}^2$: there is a cover of $\mathbb{RP}^2$ by open sets U_1, U_2, U_3 such that each preimage $q^{-1}(U_i)$ is a disjoint union $V_{i,1} \amalg V_{i,2}$ such that on each component $V_{i,j}$, the map q gives a homeomorphism $q : V_{i,j} \cong U_i$.

For any point $x \in q^{-1}(\overline{1}) = \{-1, 1\}$, we define a loop $\Gamma(x)$ at $\overline{1}$ in $\mathbb{RP}^2$ as follows: take any path α in S^2 from 1 to x. Then $\Gamma(x) = q\alpha$ is a loop in $\mathbb{RP}^2$. Note that this is well-defined **because S^2 is simply-connected**, so that any two paths between 1 and x are homotopic. When $x = 1$, this of course gives the class of the constant loop, but when $x = -1$, this gives a nontrivial loop in $\mathbb{RP}^2$. We claim that this is a bijection. So there is only one nontrivial loop!

To see this, we construct an inverse $w : \pi_1(\mathbb{RP}^2) \longrightarrow \{-1, 1\}$. We need some lemmas:

Lemma 9.2. *Given any loop in $\mathbb{RP}^2$, there is a unique lift to a path in S^2 starting at 1.*

The proof of this lemma is **exactly the same** as that of the first lemma in the proof for the circle.

Lemma 9.3. *Let $h : \gamma \simeq_p \delta$ be a path-homotopy between loops at $\overline{1}$ in $\mathbb{RP}^2$. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow S^2$ such that $\tilde{h}(0,0) = 1$.*

Again, the proof here is identical to that for the sphere. Let's see how we can use the lemmas to define w. Given any loop γ in $\mathbb{RP}^2$, there is a unique lift $\tilde{\gamma}$ in S^2 starting at 1. Since it is a lift of a loop, we must have $\tilde{\gamma}(1) \in \{-1, 1\}$. So we define $w(\gamma) = \tilde{\gamma}(1)$. That this is well-defined follows from the second lemma.

It remains to show that w really is the inverse. Let $x \in \{-1, 1\}$. Then $\Gamma(x) = q \circ \alpha$ for some path α in S^2 from 1 to x. To compute $w(\Gamma(x))$, we must find a lift of $\Gamma(x)$, but we already know that α is the lift. Thus $w(\Gamma(x)) = \alpha(1) = x$.

Similarly, suppose γ is any loop in $\mathbb{RP}^2$. Let $\tilde{\gamma}$ be a lift. Then $\Gamma(w(\gamma)) = \Gamma(\tilde{\gamma}(1)) = q\alpha$, where α is any path from 1 to $\tilde{\gamma}(1)$. But of course $\tilde{\gamma}$ is such a path and $\gamma = q\tilde{\gamma}$.

Note that we have given a bijection between $\pi_1(\mathbb{RP}^2)$ and $\{-1, 1\}$, but we have not talked about a group structure. That's because we don't need to: there is only one group of order two! We have shown that

$$\pi_1(\mathbb{RP}^2) \cong C_2.$$

10. Fri, Feb. 7

Last time, we showed that $\pi_1(\mathbb{RP}^2) \cong C_2$, the cyclic group of order two. In fact, the same proof (replacing S^2 by S^n) shows that, for $n \geq 2$, we have $\pi_1(\mathbb{RP}^n) \cong C_2$.

We will do one more example before describing the repeated phenomena we have seen in these examples. First, recall from last semester that given based spaces (X, x_0) and (Y, y_0), their **wedge sum**, or one-point union, is $X \vee Y = X \amalg Y/\sim$, where $x_0 \sim y_0$. Today, we want to study the fundamental group of $S^1 \vee S^1$ following the same approach as in the previous examples. We want to once again find a nice map $p : X \longrightarrow S^1 \vee S^1$ for some X. What we really want is an example of the following:

Definition 10.1. A surjective map $p : E \longrightarrow B$ is called a **covering map** if every $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union $p^{-1}(U) = \amalg_i V_i$ and such that p restricts to a homeomorphism $p : V_i \xrightarrow{\cong} U$. We say that the neighborhood U is **evenly covered** by p.

Remark 10.2. It is common to assume that E is connected and locally path-connected. We will assume this from now on, as it simplifies the theory. So as to avoid repeatedly saying (or writing) "connected and locally path-connected", I will simply call these spaces **very connected**.

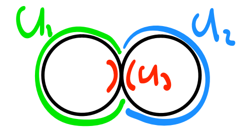
It is important to note that the neighborhood condition is local in B, not E. This contrasts with the following definition.

Definition 10.3. A map $f : X \longrightarrow Y$ is said to be a **local homeomorphism** if every $x \in X$ has a neighborhood U such that $f_{|_U} : U \xrightarrow{\cong} f(U)$ is a homeomorphism.

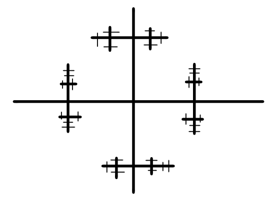
Every covering map is a local homeomorphism: given $e \in E$, take an evenly covered neighborhood U of $p(e)$. Then e is contained in one of the V_j's, which is the desired neighborhood . The converse is not true, as the next example shows.

Example 10.4. Consider the usual exponential map $p : \mathbb{R} \longrightarrow S^1$, but now restrict it to $(0, 8.123876)$. This is a local homeomorphism but not a covering map. For instance, the standard basepoint of S^1 has no evenly covered neighborhood under this map.

Ok, now back to $S^1 \vee S^1$. It is tempting to take $X = \mathbb{R}$ since $S^1 \vee S^1$ looks locally like a line, but there is a problem spot at the crossing of the figure eight. To fix this, we might try to take X to be the union of the coordinate axes inside of $\mathbb{R}^2$. This space is really just $\mathbb{R} \vee \mathbb{R}$, and so we have the map $p \vee p : \mathbb{R} \vee \mathbb{R} \longrightarrow S^1 \vee S^1$. We want a cover of $S^1 \vee S^1$ which is nicely compatible with our map from X. Suppose we consider the cover U_1, U_2, and U_3, where U_1 is the complement of the basepoint in one circle, U_2 is the complement of the basepoint in the other, and finally U_3 is some small neighborhood of the basepoint. Well, U_1 and U_2 are good neighborhoods for $p \vee p$, but U_3 is not. The map $p \vee p$ does not give a homeomorphism from each component of the preimage of U_3 to U_3. To fix this, we would want to add infinitely many cross-sections to each of the axes.



Instead, we take X to be the fractal space given in the picture. We define $p : X \longrightarrow S^1 \vee S^1$ as follows. On *horizontal* segments, use the exponential map to the *right* branch of $S^1 \vee S^1$. On *vertical* segments, use the left branch. Then the cover U_1, U_2, and U_3 from above is compatible with this new map p, and we see that p is a covering map.



Lemma 10.5. *The space X is simply-connected.*

Proof. The main point is that any loop in X is compact and therefore contained in a *finite* union of edges. Consider the edge furthest from the basepoint that contains part of the loop. The loop is homotopic to one constant on this furthest edge. This furthest edge is now no longer needed, and we have a new furthest edge. We can repeat until the loop is completely contracted. ∎