11. Mon, Feb. 10

Last time, we were working on computing $\pi_1(S^1 \vee S^1)$. We found a simply connected covering $p : X \longrightarrow S^1 \vee S^1$, where X was a fractal picture that locally looks like the coordinate axes in $\mathbb{R}^2$. Let $F = p^{-1}(*)$ be the fiber. Any point in this fiber may be uniquely described as a "word" in the letters u, r, d, and l.

Define

$$\Gamma : F \longrightarrow \pi_1(S^1 \vee S^1)$$

as follows: given $y \in F$, let α_y be any path in X from the basepoint to y. Then $\Gamma(y) = p \circ \alpha$. It does not matter which α_y we choose since X is simply-connected. We will define an inverse to Γ, but we now state the needed lemmas in the generality of coverings.

Lemma 11.1. *Let $p : E \longrightarrow B$ be a covering and suppose $p(e) = b$. Given any path starting at b in B, there is a unique lift to a path in E starting at e.*

The proof of this lemma is **exactly the same** as that of the first lemma in the proof for the circle.

Lemma 11.2. *Let $p : E \longrightarrow B$ be a covering and suppose $p(e) = b$. Let $h : \gamma \simeq_p \delta$ be a path-homotopy between paths starting at b in B. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow E$ such that $\tilde{h}(0, 0) = e$.*

Just as in the previous examples, the above lemmas allow us to define $w : \pi_1(S^1 \vee S^1) \longrightarrow F$ by the formula $w(\gamma) = \tilde{\gamma}(1)$. We will skip the verification that Γ and w are inverse, as this really follows the same script.

We have established a bijection between $\pi_1(S^1 \vee S^1)$ and the set of "words" in the letters u, r, d, and l. It remains to describe the group structure. For this, we will back up a little.

Definition 11.3. Let $p : E \longrightarrow B$ and $q : E' \longrightarrow B$ be covers of a space B. A **map of covers** from E to E' is simply a map of spaces $\varphi : E \longrightarrow E'$ such that $q \circ f = p$. These are also sometimes called **covering homomorphisms**.

The special case in which the two covers are the *same* cover and f is a homeomorphism is referred to as a **deck transformation**. We write $\mathrm{Aut}(E)$ for the set of all deck transformations of E. This is a group under composition.

Keeping our notation from earlier, let $b \in B$ be a basepoint and write $F = p^{-1}(b)$ for the fiber. Note that any deck transformation $\varphi : E \longrightarrow E$ must take F to F. Let us pick a basepoint e for E. Since we want the covering map q to be based, this means that e lies in the fiber F. We may now define a map $A : \mathrm{Aut}(E) \longrightarrow F$ by $A(\varphi) = \varphi(e)$.

Theorem 11.4. *Let $p : X \longrightarrow B$ be a covering such that X is simply connected. Then the map $A : \mathrm{Aut}(X) \longrightarrow F$ is a bijection and the composition $\Gamma \circ A$ is an isomorphism of groups $\mathrm{Aut}(X) \cong \pi_1(B)$.*

Proof. Let us first show that A is injective. Thus let φ_1 and φ_2 be deck transformations which agree at e. Let $x \in X$ be any point and let α be any path in X from e to x. Then the paths $\varphi_1 \circ \alpha$ and $\varphi_2 \circ \alpha$ are both lifts of $p \circ \alpha$ starting at the common point $\varphi_1(e) = \varphi_2(e)$. By the uniqueness of lifts, these must be the same path. It follows that their endpoints, $\varphi_1(x)$ and $\varphi_2(x)$ agree.

12. Wed, Feb. 12

It remains to show that A is surjective. Let $f \in F$ be any point in the fiber. We wish to produce a deck transformation $\varphi : X \longrightarrow X$ such that $\varphi(e) = f$. We build the map φ locally and patch together. Let $x \in X$ and pick any path $\alpha : e \rightsquigarrow x$. Then $p\alpha$ is a path in B starting at b and ending at px. By the path-lifting lemma, there is a unique lift $\widetilde{p\alpha}$ in X starting at f. We define $\varphi(x) = \widetilde{p\alpha}(1)$. From this definition, continuity is not at all clear. But the point is that since p is a covering, we can choose an evenly-covered neighborhood U of $p(x)$. Let V be the slice of $p^{-1}(U)$ containing x and V' the slice containing $\varphi(x) = \widetilde{p\alpha}(1)$. Then the restriction of φ to V is the composition of homeomorphisms

$$V \xrightarrow{p} U \xleftarrow{p} V'.$$

By construction, φ will be a covering map, as long as we can verify that it is well-defined. But if $\delta : e \rightsquigarrow x$ is another choice of path, we know that $\alpha \simeq_p \delta$ because X is simply-connected. It follows that $p\alpha \simeq_p p\delta$, and by lifting the path-homotopy, it follows that $\widetilde{p\alpha} \simeq_p \widetilde{p\delta}$, so that their right endpoints agree.

So given $f \in F$, we have built a covering map $\varphi : X \longrightarrow X$, but we wanted this to be an isomorphism. In fact, this last part comes for free if we know that it is injective. This implication will show up on homework later, so for now it is "left to the reader". We show that the φ we have built is indeed injective. Suppose $\varphi(x_1) = \varphi(x_2)$. Note that since φ is a covering map, this implies that x_1 and x_2 are in the same fiber. Let $\alpha_1 : e \rightsquigarrow x_1$ and $\alpha_2 : e \rightsquigarrow x_2$ be paths. By hypothesis, $\widetilde{p\alpha_1} = \widetilde{p\alpha_2}$. Since X is simply-connected, we know that $\widetilde{p\alpha_1} \simeq_p \widetilde{p\alpha_2}$. It follows that $p\alpha_1 \simeq_p p\alpha_2$, and it then follows, by lifting the homotopy, that $\alpha_1 \simeq_p \alpha_2$. In particular, $\alpha_1(1) = \alpha_2(1)$, so $x_1 = x_2$.

We have now established that

$$A : \mathrm{Aut}(X) \longrightarrow F$$

is a bijection. We also wanted to show that the resulting bijection $\Gamma \circ A : \mathrm{Aut}(X) \longrightarrow \pi_1(B)$ is a group isomorphism. It remains only to show that this is a group homomorphism.

Let $\varphi_1, \varphi_2 \in \mathrm{Aut}(X)$. Recall that $\Gamma(A(\varphi_1))$ is defined as follows: pick any path α_1 in X from e to $f_1 = \varphi_1(e)$. Then $\Gamma(A(\varphi_1)) = p \circ \alpha_1$. Similarly $\Gamma(A(\varphi_2)) = p \circ \alpha_2$. Now $A(\varphi_2 \circ \varphi_1) = \varphi_2 \circ \varphi_1(e) = \varphi_2(f_1)$. To compute Γ of this point, we need a path in X from e to $\varphi_2(f_1)$. But $\alpha_2 * \varphi_2(\alpha_1)$ is such a path. Then

$$\begin{aligned}\Gamma(A(\varphi_2 \circ \varphi_1)) = \Gamma(\varphi_2(f_1)) = p \circ (\alpha_2 * \varphi_2(\alpha_1)) &= p \circ \alpha_2 * p \circ \varphi_2 \circ \alpha_1 \\ &= p \circ \alpha_2 * p \circ \alpha_1 = \Gamma(A(\varphi_2) * \Gamma(A(\varphi_1)).\end{aligned}$$

■

Returning now to our example $X \longrightarrow S^1 \vee S^1$, we have identified $\pi_1(S^1 \vee S^1)$ with the group of deck transformations $X \cong X$. Any transformation can be thought of as a sequence of horizontal and vertical "moves". Writing u for an upwards shift and r for a shift to the right, any element of the group can be described by a seqence of u's, r's, and their inverses.

13. Fri, Feb. 14

Definition 13.1. A **word** in letters u, r, and their inverses is simply a sequence of these letters. We say the word is **reduced** if no u^{-1} is adjacent to a u, and similarly for the r's. The **free group** F_2 or $F(u, r)$ on the letters u and r is the set of reduced (including empty) words, where the group operation is concatenation. The inverse of any word is the same word in reversed order and with the sign of each letter reversed.

We have shown that $\pi_1(S^1 \vee S^1)$ is the free group on two letters. In particular, this is our first example of a nonabelian fundamental group.

So far, the only kind of coverings we have studied have been those in which the covering space is simply connected. Now we will relax this condition and discuss the more general theory.

Proposition 13.2. *Let $p : E \longrightarrow B$ be a covering. Then the induced map $p_* : \pi_1(E) \longrightarrow \pi_1(B)$ is injective.*

Proof. Let $\gamma \in \pi_1(E)$ and suppose $p_*(\gamma) = 0$. In other words, the loop $p \circ \gamma$ in B is null. Let $h : I \times I \longrightarrow B$ be a null-homotopy. Then this lifts to a homotopy $\tilde{h} : I \times I \longrightarrow E$ from γ (the unique lift of $p \circ \gamma$) to a lift $\tilde{c}$ of the constant loop. Since the constant loop at e is a lift of the constant loop at b, uniqueness of lifts implies that $\tilde{c}$ is the constant loop. So $\tilde{h}$ is a null-homotopy for γ. ■

Example 13.3. The only example of a covering we have discussed thus far in which the covering space is not simply connected is the n-fold cover $S^1 \longrightarrow S^1$. In this case, the cover clearly sends the generator of $\pi_1(S^1) \cong \mathbb{Z}$ to n times the generator, and the image of p_* is the subgroup $n\mathbb{Z} < \mathbb{Z}$.

Given the above result, any covering of B gives rise to a subgroup of $\pi_1(B)$. One might wonder what subgroups can arise in this way. We will see that, under mild hypotheses on B, every subgroup arises in this way.

Previously, we have studied lifting paths and path-homotopies against a covering. We can also generalize this to consider lifting arbitrary maps $f : Z \longrightarrow B$.

Proposition 13.4. *(Homotopy lifting) Let $p : E \longrightarrow B$ be a covering and $h : Z \times I \longrightarrow B$ be a homotopy between maps $f, g : Z \rightrightarrows B$. Let $\tilde{f}$ be a lift of f. Then there is a unique lift of h to $\tilde{h}$ with $\tilde{h}_0 = \tilde{f}$.*

Proposition 13.5. *(Unique lifting) Let $p : E \longrightarrow B$ be a covering and $f : Z \longrightarrow B$ a map, with Z connected. If $\tilde{f}$ and $\hat{f}$ are both lifts of f that agree at some point of Z, then they are the same lift.*

Note that in the second result, we are not asserting that a lift exists! See Theorems 8.3 and 8.4 of [Lee] for complete proofs.

Here is an alternative approach to Prop 13.4. For this discussion, we assume that B is sufficiently nice, like locally compact Hausdorff. More generally, we can work in the land of compactly generated weak Hausdorff spaces. The point is that we want mapping spaces to behave nicely.

We can think of the pair of maps $f, g : Z \rightrightarrows B$ as a pair of points of $\mathrm{Map}(Z, B)$. The homotopy h then defines a *path* in $\mathrm{Map}(Z, B)$ from f to g. Thus Prop 13.4 really just asserts that we can lift the path h uniquely once we have fixed an initial lift $\tilde{f}$. This is precisely the statement of Lemma 11.1, so we just need to check that the lemma applies. In other words, we want to know that the induced map

$$\mathrm{Map}(Z, p) : \mathrm{Map}(Z, E) \longrightarrow \mathrm{Map}(Z, B)$$

is a covering map.

$$\begin{tikzcd} & \mathrm{Map}(Z,E) \arrow[d, "\mathrm{Map}(Z,p)"] \\ I \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "h"] & \mathrm{Map}(Z,B) \end{tikzcd}$$

Actually, this does not quite work, as the map $\mathrm{Map}(Z, p)$ is not in general surjective. We write $\mathrm{Map}^{\ell}(Z, B)$ for the image of this map (we think of this as the space of "liftable" maps). On your homework, you are asked to show that the map

$$\mathrm{Map}(Z, p) : \mathrm{Map}(Z, E) \longrightarrow \mathrm{Map}^{\ell}(Z, B)$$

is a covering map. You are also asked to show that $\mathrm{Map}^{\ell}(Z, B)$ is closed and open in $\mathrm{Map}(Z, B)$. It follows that if $f \in \mathrm{Map}^{\ell}(Z, B)$, then the whole path h is contained in $\mathrm{Map}^{\ell}(Z, B)$. Thus we have reduced to a standard path-lifting problem, and Lemma 11.1 finishes the argument.

$$\begin{tikzcd} & \mathrm{Map}(Z,E) \arrow[d, "\mathrm{Map}(Z,p)"] \\ I \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "h"] & \mathrm{Map}^{\ell}(Z,B) \end{tikzcd}$$

Here is a sketch of Proposition 13.5.

Sketch. The idea is to show that the subset of Z on which the lifts agree is both open and closed; it is already given to be nonempty. For any $z \in Z$, pick an evenly-covered neighborhood U of $f(z)$. On the one hand, suppose $\tilde{f}(z) = \hat{f}(z)$. Then let V be the component of $p^{-1}(U)$ containing this point. Then $\tilde{f}^{-1}(V) \cap \hat{f}^{-1}(V)$ is a neighborhood of z on which the lifts agree (since $q : V \longrightarrow U$ is a homeomorphism).

On the other hand, if $\tilde{f}(z) \neq \hat{f}(z)$, then let $\tilde{V}$ and $\hat{V}$ be the components of $\tilde{f}(z)$ and $\hat{f}(z)$ in $p^{-1}(U)$. It follows that $\tilde{f}^{-1}(\tilde{V}) \cap \hat{f}^{-1}(\hat{V})$ is a neighborhood of z on which $\tilde{f}$ and $\hat{f}$ disagree (they land in different components of $p^{-1}(U)$). ■