17. Mon, Feb. 24

Let F be a transitive right G-set. Then $F \cong H\backslash G$ for some $H \leq G$. **We assume** that B has a universal cover $q : X \longrightarrow B$. Recall that we showed in Theorem 11.4 that the group of deck transformations of X is isomorphic to G.

Proposition 17.1. *The action of G on X via deck transformations is free and properly discontinuous.*

Proof. Let $x \in X$ and suppose $gx = x$ for some $g \in G$. Recall that here g is a covering homomorphism $X \longrightarrow X$ and thus a lift of $q : X \longrightarrow B$. By the uniqueness of lifts, since g looks like the identity at the point x, it must be the identity. This shows the action is free.

Again, let $x \in X$. We want to find a neighborhood V of x such that only finitely many translates gV meet V. Consider $b = q(x)$. Let U be an evenly-covered neighborhood of b. Then $q^{-1}(U) \cong \coprod V_i$, and $x \in Vj$ for some j. Recall that G freely permutes the pancakes V_i. In particular, the only translate of V_j that meets V_j is the identity translate eV_j. ■

According to Homework IV.4, this means that the quotient map $X \longrightarrow X/G$ is a cover. (Really, this quotient *should be* written as $G\backslash X$, but most people write it as X/G.) If we consider the action of a subgroup $H \leq G$, it is still free and properly discontinuous. So we get a covering

$$q_H : X \longrightarrow X/H = X_H$$

for every H. Moreover, the universal property of quotients gives an induced map

$$p_H : X/H \longrightarrow B.$$

18. Wed, Feb. 26

Proposition 18.1. *The map $p_H : X/H \longrightarrow B$ is a covering map, and the fiber F is isomorphic to $H\backslash G$ as a G-set.*

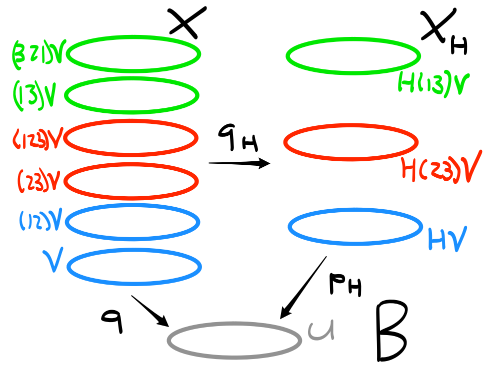
Proof. Let $b \in B$. Then we have a neighborhood U which is evenly-covered by q. Recall again that the G-action, and therefore also the H-action, simply permutes the pancakes in $p^{-1}(U)$. We thus get an action of H on the indexing set $\mathcal{I}$ for the pancakes in $p^{-1}(U)$. If we write $W_i = q_H(V_i)$, we thus have the diagram

$$\begin{array}{ccccc}
q^{-1}(U) & \xrightarrow{q_H} & p_H^{-1}(U) & \xrightarrow{p_H} & U \\
\cong \uparrow & & \cong \uparrow & & \| \\
\coprod_{i \in \mathcal{I}} V_i & \longrightarrow & \coprod_{j \in \mathcal{I}/H} W_j & \longrightarrow & U
\end{array}$$

To see that the restriction of p_H to a single W_j gives a homeomorphism, we use the fact that $q_H : V_j \longrightarrow W_j$ is a homeomorphism, since $q_H : X \longrightarrow X_H$ is a covering, and that $q : V_j \longrightarrow U$ is a homeomorphism. It follows that $p_H = q \circ q_H^{-1}$ is a homeomorphism.

For the identification of the fiber $F \subseteq X_H$, notice that the H-action on X acts on each fiber separately, and the quotient of this action on the fiber of X gives precisely $H\backslash G$. ■

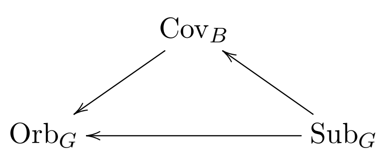
Example 18.2. Suppose that $G = \Sigma_3$, the symmetric group on 3 letters, and let $H = \{e, (12)\} \leq G$. If we take an evenly-covered neighborhood U in B, then the situation described in the proof above is given in the picture to the right.
As an aside, note that X_H here is an example of a covering in which the deck transformations do *not* act transitively on the fibers.



To sum up, we have shown that *if* B has a universal cover, then the assignment $(E,p) \mapsto F$ gives an "equivalence of categories" between coverings of B (Cov_B) and G-orbits (Orb_G). We can also relate both of these "categories" to subgroups of G in the following way.

We consider subgroups of G (Sub_G), and given any two subgroups $H, K \leq G$, we consider homomorphisms $H \longrightarrow K$ that are induced by conjugation by an element $\gamma \in G$. Given $H \in \text{Sub}_G$, we get a G-orbit $H\backslash G$. Given a homomorphism $\text{con}_\gamma : H \longrightarrow K$, we get an induced map $H\backslash G \longrightarrow K\backslash G$ defined by $Hg \mapsto K\gamma g$. We have seen (Prop 16.3) that every map between these orbits is of this form.

We can also relate Sub_G to Cov_B in the following way. Given a subgroup H, we have already constructed the covering X_H (assuming X exists). And we know, by passing through Orb_G, that conjugation homomorphisms $H \longrightarrow K$ are in bijective correspondence with maps of covering $X_H \longrightarrow X_K$. We thus now have equivalences of categories as in the diagram below



The last result we need to tie this story together is the existence of universal covers.

Definition 18.3. Let B be any space. A subset $U \subseteq B$ is **relatively simply connected** (in B) if every loop in U is contractible in B. We say that B is **semilocally simply connected** if every point has a relatively simply connected neighborhood.

Theorem 18.4. *Let B be very connected. Then there exists a universal cover $X \longrightarrow B$ if and only if B is semilocally simply connected.*

Proof. For convenience, we fix a basepoint $b_0 \in B$.

We start by working backwards. That is, suppose that $q : X \longrightarrow B$ exists. Given a point $b \in B$, what can we say about the fiber $q^{-1}(b)$? Pick a basepoint $x_0 \in q^{-1}(b_0)$. Then, for each $f \in q^{-1}(b)$, we get a (unique) path-homotopy class of paths $\alpha : x_0 \rightsquigarrow f$. Composing with the covering map q gives a (unique) path-homotopy class of paths $q \circ \alpha : b_0 \rightsquigarrow b$. This now gives a description of the fiber $q^{-1}(b)$ purely in terms of B.

We now take this as a starting point. As a set, we take X to be the set of path-homotopy classes of paths starting at b_0. The map $q : X \longrightarrow B$ takes a class $[\gamma]$ to the endpoint $\gamma(1)$. It remains to (1) topologize X, (2) show that q is a covering map, and (3) show that X is simply-connected.

19. Fri, Feb. 28

We specify the topology on X by giving a basis. Let γ be a path in B starting at b_0. Let U be any relatively simply-connected neighborhood of the endpoint $\gamma(1)$. Define a subset $U[\gamma] \subseteq X$

to be the set of equivalence classes of paths of the form $[\gamma\delta]$, where $\delta : I \longrightarrow U$ is a path in U. These cover X since each $[\gamma]$ is contained in some $U[\gamma]$, since B is semilocally simply-connected. Now suppose that $\gamma \in U_1[\gamma_1] \cap U_2[\gamma_2]$. Then the intersection $U = U_1 \cap U_2$ of two relatively simply connected subsets of B is again relatively simply connected. Thus

$$\gamma \in U[\gamma] \subseteq U_1[\gamma_1] \cap U_2[\gamma_2].$$

We have shown that the $U[\gamma]$ give a basis for a topology on X.

Next, we show that q is continous. Let $V \subseteq B$ be open and let $q([\gamma]) \in V$, so that $\gamma(1) \in V$. Then we can find a relatively simply connected U satisfying $\gamma(1) \in U \subseteq V$. So $U[\gamma]$ is a neighborhood of $[\gamma]$ in $q^{-1}(V)$, as desired.

Since B is path-connected, it follows that q is surjective. Let $b \in B$ and let $b \in U$ be a path-connected, relatively simply-connected neighborhood. We claim that U is evenly covered by q. First, we claim that

$$q^{-1}(U) = \bigcup_{[\gamma] \in q^{-1}(b)} U[\gamma].$$

It is clear that the RHS is contained in the LHS. Suppose that $q([\alpha]) \subseteq U$. Then $\alpha(1) \in U$ and we may pick a path $\delta : \alpha(1) \rightsquigarrow b$ in U. Then $\alpha \in U[\alpha\delta]$. By the definition of the topology on X, each $U[\gamma]$ is open. Finally, we wish to show that this is a disjoint union. Thus suppose that $[\alpha] \in U[\gamma_1] \cap U[\gamma_2]$. This means that

$$[\alpha] = [\gamma_1\delta_1] = [\gamma_2\delta_2].$$

In other words,

$$[\gamma_1\delta_1\overline{\delta_2}] = [\gamma_2].$$

Since U is relatively simply-connected, this implies that $[\gamma_1] = [\gamma_2]$. So any two overlapping $U[\gamma]$ are in fact the same. To finish the proof that q is a covering, we need to show that q restricts to a homeomorphism $q : U[\gamma] \xrightarrow{\cong} U$. Surjectivity follows from the assumption that U is path-connected. Injectivity is the relatively simply-connected hypothesis. Finally, q takes any basis $V[\lambda]$ to the open set V, so it is open. We have shown that q is a covering map.

The final step is to show that X is very connected and simply connected. It is clear that X is locally path-connected. Next, we show that X is path-connected (and therefore connected). Let $[\gamma] \in X$. We define a path h in X from the constant path $[c_{b_0}]$ to $[\gamma]$ by $h(s) = [\gamma|_{[0,s]}]$. In the interest of time, we skip the verification that h is continuous (but see Lee, proof of Theorem 11.43).

To see that X is simply connected, let Γ be a loop in X at the basepoint $[c_{b_0}]$. Write $\gamma = q \circ \Gamma$. Then Γ is a lift of γ, but so is the loop $s \mapsto [\gamma_{[0,s]}]$. By uniqueness of lifts, $[\Gamma(s)] = [\gamma_{[0,s]}]$. Then, since Γ is a loop, we have

$$[\gamma] = [\gamma_{[0,1]}] = [\Gamma(1)] = [\Gamma(0)] = [\gamma_{[0,0]}] = [c_{b_0}].$$

In other words, γ is null. Since q is a covering, this implies that Γ is null as well. ∎