20. Mon, Mar. 3

Dashing through the snow, in a one-horse open sleigh...

21. Wed, Mar. 5

Exam Day!!

Long time the manxome foe he sought—
So rested he by the Tumtum tree,
And stood awhile in thought.

22. Fri, Mar. 7

Last time (week), we showed that if a space is **semilocally simply-connected**, then it has a universal cover. So to provide an example of a space without a universal cover, it suffices to give an example of a space with a point which has no relatively simply connected neighborhood.

Example 22.1 (The Hawaiian earring)**.** Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0)$. So each such circle is tangent to the origin. Let $C = \cup_n C_n$. We claim that the origin has no relatively simply connected neighborhood. Indeed, let U be any neighborhood of the origin. Then for large enough n, the circle C_n is contained in U. A loop α that goes once around the circle C_n is not contractible in C. To see this, note that the map $r_n : C \longrightarrow S^1$ which collapses every circle except for C_n is a retraction. The loop $r \circ \alpha$ is not null, so α can't be null.

This example looks like an infinite wedge of circles, but it is not just a wedge. For instance, in each C_n consider an open interval U_n of radian length $1/n$ centered at the origin. The union $U = \cup_n U_n$ of the U_n's is open in the infinite wedge of circles but not in C, since no ϵ-neighborhood of the origin is contained in U.

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The focus of the next unit of the course will be on computation of fundamental groups.

One example we have already studied is the fundamental group of $S^1 \vee S^1$. We saw that this is the free group on two generators. We will see similarly that the fundamental group of $S^1 \vee S^1 \vee S^1$ is a free group on three generators (the generators are the loops around each circle). We will also want to compute the fundamental group of the two-holed torus (genus two surface), the Klein bottle, and more.

The main idea will be to decompose a space X into smaller pieces whose fundamental groups are easier to understand. For instance, if $X = U \cup V$ and we understand $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$, we might hope to recover $\pi_1(X)$.

Proposition 22.2. *Suppose that $X = U \cup V$, were U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then the smallest subgroup of $\pi_1(X)$ containing the images of both $\pi_1(U)$ and $\pi_1(V)$ is $\pi_1(X)$ itself.*

In group theory, we would say $\pi_1(X) = \pi_1(U)\pi_1(V)$.

Note that we really do need the assumption that $U \cap V$ is path-connected. If we consider U and V to be open arcs that together cover S^1, then both U and V are simply-connected, but their intersection is not path-connected. Note that here that the product of two trivial subgroups is not $\pi_1(S^1) \cong \mathbb{Z}$!

Proof. Let $\gamma : I \longrightarrow X$ be a loop at x_0. By the Lebesgue number lemma, we can subdivide the interval I into smaller intervals $[s_i, s_{i+1}]$ such that each subinterval is taken by γ into either U or V. We write γ_1 for the restriction of γ to the first subinterval. Suppose, for the sake of argument, that γ_1 is a path in U and that γ_2 is a path in V. Since $U \cap V$ is path-connected, there is a path δ_1 from $\gamma_1(1)$ to x_0. We may do this for each γ_i. Then we have

$$[\gamma] = [\gamma_1] * [\gamma_2] * [\gamma_3] * \cdots * [\gamma_n] = [\gamma_1 * \delta_1] * [\delta_1^{-1} * \gamma_2 * \delta_2] * \cdots * [\delta_{n-1}^{-1} * \gamma_n]$$

This expresses the loop γ as a product of loops in U and loops in V. ∎

This is a start, but it is not the most convenient formulation. In particular, if we would like to use this to calculate $\pi_1(X)$, then thinking of the product of $\pi_1(U)$ and $\pi_1(V)$ *inside of* $\pi_1(X)$ is not so helpful. Rather, we would like to express this in terms of some external group defined in terms of $\pi_1(U)$ and $\pi_1(V)$. We have homomorphisms

$$\pi_1(U) \longrightarrow \pi_1(X), \qquad \pi_1(V) \longrightarrow \pi_1(X),$$

and we would like to put these together to produce a map from some sort of product of $\pi_1(U)$ and $\pi_1(V)$ to $\pi_1(X)$. Could this be the direct product $\pi_1(U) \times \pi_1(V)$? No. Elements of $\pi_1(U)$ commute with elements of $\pi_1(V)$ in the product $\pi_1(U) \times \pi_1(V)$, so this would also be true in the image of any homomorphism $\pi_1(U) \times \pi_1(V) \longrightarrow \pi_1(X)$.

What we want instead is a group freely built out of $\pi_1(U)$ and $\pi_1(V)$. The answer is the **free product** $\pi_1(U) * \pi_1(V)$ of $\pi_1(U)$ and $\pi_1(V)$. Its elements are finite length words $g_1 g_2 g_3 g_4 \dots g_n$, where each g_i is in either $\pi_1(U)$ or in $\pi_1(V)$. Really, we use the reduced words, where none of the g_i is allowed to be an identity element and where if $g_i \in \pi_1(U)$ then $g_{i+1} \in \pi_1(V)$.

Example 22.3. We have already seen an example of a free product. The free group F_2 is the free product $\mathbb{Z} * \mathbb{Z}$.

Example 22.4. Similarly, the free group F_3 on three letters is the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Example 22.5. Let C_2 be the cyclic group of order two. Then the free product $C_2 * C_2$ is an infinite group. If we denote the nonidentity elements of the two copies of C_2 by a and b, then elements of $C_2 * C_2$ look like a, ab, $ababa$, $ababababa$, $bababa$, etc.

Note that there is a homomorphism $C_2 * C_2 \longrightarrow C_2$ that sends both a and b to the nontrivial element. The kernel of this map is all words of even length. This is the (infinite) subgroup generated by the word ab (note that $ba = (ab)^{-1}$). In other words, $C_2 * C_2$ is an extension of C_2 by the infinite cyclic group $\mathbb{Z}$. Another way to say this is that $C_2 * C_2$ is a semidirect product of C_2 with $\mathbb{Z}$.