23. Mon, Mar. 10

The free product has a universal property, which should remind you of the property of the disjoint union of spaces $X \amalg Y$. First, for any groups H and K, there are inclusion homomorphisms $H \longrightarrow H * K$ and $K \longrightarrow H * K$.

Proposition 23.1. *Suppose that G is any group with homomorphisms $\varphi_H : H \longrightarrow G$ and $\varphi_K : K \longrightarrow G$. Then there is a (unique) homomorphism $\Phi : H * K \longrightarrow G$ which restricts to the given homomorphisms from H and K.*

Our result from last time can be restated as follows:

Proposition 23.2. *Suppose that $X = U \cup V$, where U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then the natural homomorphism*

$$\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$$

is surjective.

Now that we have a surjective homomorphism to $\pi_1(X)$, the next step is to understand the kernel N. Indeed, then the First Isomorphism Theorem will tell us that $\pi_1(X) \cong \pi_1(U) * \pi_1(V)/N$. Here is one way to produce an element of the kernel. Consider a loop α in $U \cap V$. We can then consider its image $\alpha_U \in \pi_1(U)$ and $\alpha_V \in \pi_1(V)$. Certainly these map to the same element of $\pi_1(X)$, so $\alpha_U \alpha_V^{-1}$ is in the kernel.

Proposition 23.3. *With the same assumptions as above, the kernel K of $\pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$ is the normal subgroup N generated by elements of the form $\alpha_U \alpha_V^{-1}$.*

Recall that the normal subgroup generated by the elements $\alpha_U \alpha_V^{-1}$ can be characterized either as (1) the intersection of all normal subgroups containing the $\alpha_U \alpha_V^{-1}$ or (2) the subgroup generated by all conjugates $g \alpha_U \alpha_V^{-1} g^{-1}$.

Proof. Again, it is clear that the kernel K must contain the subgroup N. It remains to show that $K \leq N$. Consider an element of K. For simplicity, we assume it is $\alpha_1 \cdot \beta_1 \cdot \alpha_2$, where $\alpha_i \in \pi_1(U)$ and $\beta_1 \in \pi_1(V)$. The assumption that this is in K means that there exists a homotopy $H : I \times I \longrightarrow X$ from the path composition $\alpha_1 * \beta_1 * \alpha_2$ in X to the constant loop.

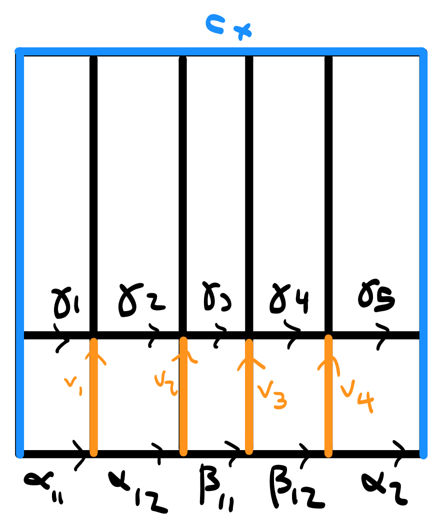
24. Wed, Mar. 12

By the Lebesgue lemma, we may subdivide the square into smaller squares such that each small square is taken by H into either U or V. Again, we suppose for simplicity that this divides α_1 into α_{11} and α_{12} and β_1 into β_{11} and β_{12} (and α_2 is not subdivided).

Note that we cannot write

$$\alpha_1 \cdot \beta_1 \cdot \alpha_2 = \alpha_{11} \cdot \alpha_{12} \cdot \beta_{11} \cdot \beta_{12} \cdot \alpha_2$$

in $\pi_1(U) * \pi_1(V)$ since these are not all loops. But we can fix this, using the same technique as in the proof of Prop 22.2. In other words, we append a path δ back to x_0 at the end of every path on an edge of a square. If that path is in U (or V or $U \cap V$), we take δ in U (or V or $U \cap V$). Also, if the path already begins or ends at x_0, we do not append a δ. For convenience, we keep the same notation, but remember that we have really converted all of these paths to loops.



Let us turn our attention now to the homotopy H on the first (lower-left) square. Either H takes this into U or into V. If it is U, then we get a path homotopy in U $\alpha_{11} \simeq_p \gamma_1 \cdot v_1^{-1}$. If, on the other hand, H takes this into V, then it follows that α_{11} is really in $U \cap V$. This gives us a path homotopy in V $\alpha_{11} \simeq_p \gamma_1 \cdot v_1^{-1}$. But the group element α_{11} comes from $\pi_1(U)$ in the free product $\pi_1(U) * \pi_1(V)$. We would like to replace this with the element α_{11} from $\pi_1(V)$.

Lemma 24.1. *Let γ be any loop in $U \cap V$. Then, in the quotient group $Q = \big(\pi_1(U) * \pi_1(V)\big)/N$, the elements γ_U and γ_V are equivalent.*

Proof. The point is that

$$\gamma_V N = \gamma_U \gamma_U^{-1} \gamma_V N = \gamma_U \cdot \Big((\gamma^{-1})_U (\gamma^{-1})_V^{-1}\Big) N = \gamma_U N.$$

■

From here on out, we work in the quotient group Q. The goal is to show that the original element $\alpha_1 \cdot \beta_1 \cdot \alpha_2$ is trivial in Q. According to the above, we can replace $(\alpha_1)_U(\beta_1)_V(\alpha_2)_U$ with either

$$(\gamma_1)_U (v_1^{-1})_U (\alpha_{12})_U (\beta_{11})_V (\beta_{12})_V (\alpha_2)_U$$

or

$$(\gamma_1)_V (v_1^{-1})_V (\alpha_{12})_U (\beta_{11})_V (\beta_{12})_V (\alpha_2)_U.$$

We then do the same with each of α_{12}, ..., α_2. The resulting expression will have adjacent terms v_i and v_i^{-1}. For the same i, these two loops may have the same label (U or V) or different labels. But by the lemma, we can always change the label if the loop lies in the intersection. So we get the path-composition of the paths along the top edges of the bottom squares. We then repeat the procedure, moving up rows until we get to the very top. But of course the top edges of the top squares are all constant loops. It follows that we end up with the trivial element (of Q). So $K = N$. ■

There is another, more elegant, way to state the Van Kampen theorem.

Definition 24.2. Suppose given a pair of group homomorphisms $\varphi_G : H \longrightarrow G$ and $\varphi_K : H \longrightarrow K$. We define the **amalgamated free product** (or simply amalgamated product) to be the quotient

$$G *_H K = G * K/N,$$

where $N \trianglelefteq G * K$ is the normal subgroup generated by elements of the form $\varphi_G(h)\varphi_K(h)^{-1}$.

It is easy to check that *the amalgamated free product satisfies the universal property of the pushout in the category of groups.*

Theorem 24.3 (Van Kampen, restated)**.** . *Let X be given as a union of two open, path-connected subsets U and V with path-connected intersection $U \cap V$. Then the inclusions of U and V into X induce an isomorphism*

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \xrightarrow{\cong} \pi_1(X).$$

Since the pasting lemma tells us that in this situation, X can itself be written as a pushout, the Van Kampen theorem can be interpreted as the statement that, under the given assumptions, the fundamental group construction takes a pushout of spaces to a pushout of groups.

One important special case of this result is when $U \cap V$ is simply connected.

Example 24.4. Take $X = S^1 \vee S^1$. Take U and V to be neighborhoods of the two circles, so that the intersection $U \cap V$ looks like an 'X'. Then $U \cap V$ is contractible, and U and V are both equivalent to S^1. We conclude from this that

$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} \cong F_2.$$

Example 24.5. Take $X = S^1 \vee S^1 \vee S^1$. We can take U to be a neighborhood of $S^1 \vee S^1$ and V to be a neighborhood of the remaining S^1. Then

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong (\mathbb{Z} * \mathbb{Z}) * \mathbb{Z} \cong F_3.$$

25. Fri, Mar. 14

A natural question now is whether $\pi_1(X \vee Y)$ is always the free product of the $\pi_1(X)$ and $\pi_1(Y)$. Not quite, but a mild assumption allows us to make the conclusion. Note that in the $S^1 \vee S^1$ example, we needed to know that the neighborhoods U and V were homotopy equivalent to S^1 (and that the intersection was contractible).

Definition 25.1. We say that $x_0 \in X$ is a **nondegenerate basepoint** for X if x_0 has a neighborhood U such that x_0 is a deformation retract of U.

Proposition 25.2. *Let x_0 and y_0 be nondegenerate basepoints for X and Y, respectively. Then*

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

Proof. Suppose that x_0 is a deformation retract of the neighborhood $N_X \subseteq X$ and that y_0 is a deformation retract of the neighborhood $N_Y \subseteq Y$. Let $U = X \vee N_Y$ and $V = N_X \vee Y$. Then $U \cap V = N_X \vee N_Y$. The retracting homotopies for N_X and N_Y give $U \simeq X$, $V \simeq Y$, and $U \cap V \simeq *$. The van Kampen theorem then gives the conclusion. ■

Aside. Last time, I mentioned an example of a group G and subgroup H which was conjugate to a proper subgroup of itself. I want to observe another consequence of this example. The group G was

$$G = \langle a, b \mid bab^{-1} = a^2 \rangle,$$

and the subgroup H was $\langle a \rangle$. Then clearly $bHb^{-1} = \langle a^2 \rangle < H$. This group G is called a **Baumslag-Solitar** group.

Note that the map $\phi : H\backslash G \longrightarrow H\backslash G$ defined by $\phi(Hg) = Hbg$ is well-defined since $bHb^{-1} < H$. This function is necessarily surjective, since $H\backslash G$ is a transitive G-set. But I claim it is not injective. To see this, first note that

$$\phi(Hb^{-1}a) = Hbb^{-1}a = Ha = H = \phi(Hb^{-1}).$$

Next, we can see that $Hb^{-1}a \neq Hb^{-1}$ or, equivalently, $Hb^{-1}ab \neq H$. This holds because $b^{-1}ab$ is a square root of a and is not in H. Thus ϕ is a self-map of $H\backslash G$ which is not an isomorphism.

We can then transport this statement about G-orbits to a statement about coverings. Let X be a space equipped with a free, properly discontinuous G-action, and let $B = G\backslash X$. As usual, we let $X_H = H\backslash X$, which then covers B. We know that the map ϕ described above corresponds to a covering transformation $X_H \longrightarrow X_H$, but in this case it is not a homeomorphism.