Math 751 - Topics in Topology
Homework 5
Spring 2015

1. Let $F \to E \xrightarrow{p} B$ be a fibration. Suppose that B is path-connected and that $\pi_1(B)$ acts trivially on $\mathrm{H}_*(F)$. Let $\mathbb{F}$ be a field, and assume that $\mathrm{H}_*(B;\mathbb{F})$ and $\mathrm{H}_*(F;\mathbb{F})$ are both finite-dimensional. Define the $\mathbb{F}$-Euler characteristic of a space X by the formula

$$\chi_{\mathbb{F}}(X) := \sum_i (-1)^i \dim_{\mathbb{F}} \mathrm{H}_i(X;\mathbb{F}).$$

Show that $\mathrm{H}_*(E;\mathbb{F})$ is finite-dimensional and that

$$\chi_{\mathbb{F}}(E) = \chi_{\mathbb{F}}(B) \cdot \chi_{\mathbb{F}}(F).$$

2. Let $n \geq 1$.

   (a) We have, as discussed previously, an action of S^1 on S^{2n+1}. The subgroup inclusion $\mathbb{Z}/2\mathbb{Z} \cong S^0 \subseteq S^1$ gives a fiber sequence

$$S^1/\mathbb{Z}/2\mathbb{Z} \longrightarrow S^{2n+1}/\mathbb{Z}/2\mathbb{Z} \longrightarrow S^{2n+1}/S^1,$$

   which we can rewrite as a fiber sequence

$$S^1 \longrightarrow \mathbb{RP}^{2n+1} \longrightarrow \mathbb{CP}^n.$$

   Use the Serre spectral sequence to determine the cohomology ring $\mathrm{H}^*(\mathbb{RP}^{2n+1};\mathbb{Z})$.

   (b) Use part (a) and the inclusion $\mathbb{RP}^{2n} \hookrightarrow \mathbb{RP}^{2n+1}$ to determine the cohomology ring $\mathrm{H}^*(\mathbb{RP}^{2n};\mathbb{Z})$.

   (c) Use parts (a) and (b) to determine $\mathrm{H}^*(\mathbb{RP}^\infty;\mathbb{Z})$.

3. (a) Use the path-loop fibration $\Omega(S^n) \longrightarrow P_*(S^n) \longrightarrow S^n$ to determine the cohomology ring $\mathrm{H}^*(\Omega S^n;\mathbb{Q})$. (Hint: Your answer should depend on the parity of n).

   (b) $(\star)$ Determine the integral cohomology ring $\mathrm{H}^*(\Omega S^n;\mathbb{Z})$.

4. For any n, denote by $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ the map $\lambda(x_1,\ldots,x_n) = (0, x_1, \ldots, x_n)$. For any $k \leq n$, define a map $\Lambda : V_k(\mathbb{R}^n) \longrightarrow V_{k+1}(\mathbb{R}^{n+1})$ by

$$(\mathbf{v}_1,\ldots,\mathbf{v}_k) \mapsto (\mathbf{e}_1, \lambda(\mathbf{v}_1), \ldots, \lambda(\mathbf{v}_k)).$$

Also denote by $p_k : V_{k+1}(\mathbb{R}^n) \longrightarrow V_k(\mathbb{R}^n)$ the fibration

$$p_k(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}) = (\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

The maps Λ assemble together to yields maps of fiber sequences

$$\begin{tikzcd}
S^{n-k} \arrow[r] \arrow[d, equal] & V_j(\mathbb{R}^{n-k+j}) \arrow[r, "p_j"] \arrow[d, "\Lambda"] & V_{j-1}(\mathbb{R}^{n-k+j}) \arrow[d, "\Lambda"] \\
S^{n-k} \arrow[r] & V_{j+1}(\mathbb{R}^{n-k+j+1}) \arrow[r, "p_{j+1}"'] & V_j(\mathbb{R}^{n-k+j+1})
\end{tikzcd}$$

Use induction on j (with base case $j = 2$) to show that $\mathrm{H}^*(V_k(\mathbb{R}^n); \mathbf{F}_2)$ has a simple system of generators $\{x_{n-k}, \ldots, x_{n-1}\}$, where $\deg(x_i) = i$.