Math 752 - Hopf algebras
Homework 3
Spring 2017

1. Let $\mathcal{P}$ be a finite poset and R a commutative ring. Denote by $I(\mathcal{P})$ the set of intervals in $\mathcal{P}$, in other words the set of pairs (a, b) where $a \leq b$. If f and g are functions $I(\mathcal{P}) \longrightarrow R$, define their convolution by the formula

$$(f * g)(a, b) = \sum_{a \leq x \leq b} f(a, x) g(x, b).$$

   (a) Show that the function $\delta(a, b) := \begin{cases} 1 & a = b \\ 0 & \text{else} \end{cases}$ is the identity element in the convolution algebra.

   (b) Denote by $\zeta : I(\mathcal{P}) \longrightarrow R$ the constant function at 1. Define the Möbius function μ recursively by $\mu(x, x) = 1$ and $\mu(x, y) := -\sum_{x \leq z < y} \mu(x, z)$ if $x < y$. Show that μ is inverse to ζ in the convolution algebra.

   (c) (Möbius inversion) Suppose that $\{M_x \mid x \in \mathcal{P}\}$ is a basis for V and define a new basis by $F_x := \sum_{y \geq x} M_y$. Show that $\{M_x\}$ can be recovered from $\{F_y\}$ by the formula

$$M_x = \sum_{y \geq x} \mu(x, y) F_y.$$

2. Show that, in the map $\mathcal{R} : \mathcal{N}\mathrm{Sym} \longrightarrow \mathcal{T}$ of Proposition 15.1, the assignment $\mathcal{R}(H_n) = R_n$ preserves coproducts.

3. The **nilpotence height** of an element $x \in \mathcal{A}$ is the smallest natural number k such that $x^k = 0$. Find the nilpotence height for Sq^i, for $i = 1, \ldots, 7$.