CLASS NOTES
MATH 752 (SPRING 2017)

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CONTENTS

$\boxed{\textbf{Wed, Jan. 11}}$

This will be a course on Hopf Algebras, and one of our main guiding examples will be group algebra $k\{G\}$ on a group G (k is a field and G is group). Recall that, as a k-vector space, this can be defined as the vector space with basis the elements of G. The multiplication in G turns $k\{G\}$ into a k-algebra (as we will review later). But this has extra structure. For example, the inverse in G induces a map $k\{G\} \longrightarrow k\{G\}$, and a Hopf algebra is essentially an algebra with the extra structure that we see in a group algebra.

Many topics from the study of groups generalize to the setting of Hopf algebras. Among the topics we aim to study include

- sub-Hopf algebras (analogous to subgroups)
- quotients by "normal" sub-Hopf algebras
- extensions of Hopf algebras
- Hopf algebra cohomology

Time permitting, we will see at the end of the course how these concepts help us study homotopy groups of spheres.

Date: April 28, 2017.

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1. BIALGEBRAS

1.1. **Coalgebras.** Let S be a set, and R a field (or really any commutative ring). If we denote by R^S the set of R-valued functions on S, then pointwise addition and multiplication makes R^S into a commutative ring. Now, for fixed R, the assignment $X \mapsto S^X$ defines a functor

$$\mathbf{Set}^{op} \longrightarrow \mathrm{CRing},$$

which means in particular that for any function $f : X \longrightarrow Y$, the function

$$R^Y \xrightarrow{f^*} R^X$$
$$\varphi \mapsto \varphi \circ f$$

is a ring homomorphism.

Example 1.1. In topology, this construction arises as follows: Universal Coefficients gives an identification

$$\mathrm{H}^0(X; R) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_0(X;\mathbb{Z}), R) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\{\pi_0(X)\}, R) \cong \mathrm{Hom}_{\mathrm{Set}}(\pi_0(X), R) = R^{\pi_0(X)}.$$

Now suppose that S in addition has a multiplication $S \times S \xrightarrow{\mu} S$. This induces a ring homomorphism

$$\mu^* : R^S \longrightarrow R^{S\times S}.$$

For sets S and T, we define $\Phi : R^S \otimes R^T \longrightarrow R^{S\times T}$ by $f \otimes g \mapsto f \cdot g$, where $(f \cdot g)(s, t) = f(s) \cdot g(t)$. [Recall that in the setting of commutative rings, tensor product is the coproduct. So the ring homomorphism Φ is uniquely specified by the two ring homomorphisms $p_1^* : R^S \longrightarrow R^{S\times S}$ and $p_2^* : R^S \longrightarrow R^{S\times S}$.]

Proposition 1.2. *For finite sets S and T, the above map $\Phi : R^S \otimes R^T \longrightarrow R^{S\times T}$ is an isomorphism.*

Proof. These are both finite free R modules of rank $|S \times T| = |S| \cdot |T|$.

Since S is finite, R^S has basis given by the characteristic functinos $\{s^*\}_{s\in S}$, where

$$s^*(t) = \begin{cases} 1 & t = s \\ 0 & \text{else.} \end{cases}$$

Then Φ identifies the basis $\{s^* \otimes t^*\}$ for $R^S \otimes R^T$ with the basis $\{(s, t)^*\}$ for $R^{S\times T}$. ∎

Remark 1.3. Actually, only one of S and T is required to be finite, as we will see next time.

This allows us to express the above map as a ring homomorphism (again, assuming S finite).

$$\Delta = \mu^* : R^S \longrightarrow R^S \otimes_R R^S.$$

It is worth being explicit about this map. It suffices to specify Δ on the basis elements, and the formula is

$$\Delta(s^*) = \sum_{t\cdot t'=s} t^* \otimes t'^*.$$

Thus for a general element of R^S written in terms of the basis as $f = \sum_s r_s s^*$, the formula is

$$\Delta(\sum_r r_s s^*) = \sum_{t,t'} r_{t\cdot t'}\, t^* \otimes t'^*.$$

Example 1.4. If M is a (finite) monoid with unit element $e \in M$, then evaluating at $e \in M$ gives a homomorphism $\varepsilon : R^M \longrightarrow R$. That e is a unit on the left means that the composite

$$M \cong \{e\} \times M \hookrightarrow M \times M \xrightarrow{m} M$$

is the identity. Applying (contravariant) functoriality gives that

$$R^M \xrightarrow{\Delta} R^M \otimes_R R^M \xrightarrow{\varepsilon \otimes \mathrm{id}} R \otimes_R R^M \cong R^M$$

is the identity. Similarly, we see that ε satifies a right unit-like law. Moreover, the associativity of m implies that the two composites

$$\begin{tikzcd} R^M \arrow[r, "\Delta"] & R^M \otimes R^M \arrow[r, shift left, "\Delta \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes \Delta"'] & R^M \otimes R^M \otimes R^M \end{tikzcd}$$

agree.

This is an example of the following

Definition 1.5. A (coassociative) coalgebra is an R-module C equipped with R-module maps $C \xrightarrow{\varepsilon} R$ (the *counit*) and $C \xrightarrow{\Delta} C \otimes C$ (the *comultiplication*) satisfying counit laws on both sides as well as a coassociative law.

Fri, Jan. 13

Example 1.6. Consider the finite monoid (in fact group) $M = C_2 = \{e, \sigma\}$. Then R^{C_2} has basis $\{e^*, \sigma^*\}$. The counit is $\varepsilon(e^*) = 1$ and $\varepsilon(\sigma^*) = 0$, and

$$\Delta(e^*) = e^* \otimes e^* + \sigma^* \otimes \sigma^*, \qquad \Delta(\sigma^*) = \sigma^* \otimes e^* + e^* \otimes \sigma^*.$$

Example 1.7. Coalgebras also show up in algebraic topology. If X is a topological space and R is a field (or more generally a PID, such as $\mathbb{Z}$), then the Kunneth theorem gives an isomorphism $\mathrm{H}_0(X \times X; R) \cong \mathrm{H}_0(X; R) \otimes_R \mathrm{H}_0(X; R)$. Thus the diagonal map on X specifies an R-module homomorphism

$$\mathrm{H}_0(X; R) \xrightarrow{(\Delta_X)_*} \mathrm{H}_0(X \times X; R) \cong \mathrm{H}_0(X; R) \otimes_R \mathrm{H}_0(X; R).$$

The counit $\mathrm{H}_0(X; R) \longrightarrow \mathrm{H}_0(*; R) \cong R$ is induced by the collapse map $X \longrightarrow *$. The coalgebra laws then follow from functoriality of $\mathrm{H}_0(-; R)$, once you check the relevant compatibilities of the space-level maps. For example, commutativity of the triangle

$$\begin{tikzcd} & X \\ X \arrow[ur, equal] \arrow[r, "\Delta"] & X \times X \arrow[u, "\mathrm{id} \times c"'] \end{tikzcd}$$

shows that the counit map satisfies one of the two counit laws.

We saw the R-algebra $\mathrm{H}^0(X; R)$ appear last time, and Universal coefficients tells us this is the R-linear dual of the coalgebra $\mathrm{H}_0(X; R)$. This is an example of the following phenomenon:

Proposition 1.8. *Let C be an R-coalgebra. Then $A := \mathrm{Hom}_R(C, R)$ inherits the structure of an R-algebra.*

Proof. The multiplication is

$$A \otimes_R A = \mathrm{Hom}_R(C, R) \otimes_R \mathrm{Hom}_R(C, R) \xrightarrow{\otimes} \mathrm{Hom}_R(C \otimes_R C, R \otimes_R R) \xrightarrow{\Delta^*} \mathrm{Hom}_R(C, R) = A.$$

The unit is $R = \mathrm{Hom}_R(R, R) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(C, R) = A$. The unit and associativity laws are then straightforward to check from the coalgebra laws for C. ■

Note that we did not need any finiteness hypotheses. What if we dualize an algebra? For this, we will need

Lemma 1.9. *If M and N are R-modules, and either one is free of finite rank, then then map*

$$\mathrm{Hom}_R(M,R) \otimes \mathrm{Hom}_R(N,R) \longrightarrow \mathrm{Hom}_R(M \otimes N, R)$$

specified by $\lambda \otimes \varphi \mapsto \lambda \cdot \varphi$ is an isomorphism.

Proof. Suppose that $M \cong R\{S\} \cong \bigoplus_S R$. Then, since direct sums distribute over tensor products and since $\mathrm{Hom}_R(R,R) \cong R$, the left hand side becomes $\bigoplus_S \mathrm{Hom}_R(N,R)$, whereas the right hand side gives $\mathrm{Hom}_R(\bigoplus_S N, R) \cong \bigoplus_S \mathrm{Hom}_R(N,R)$. ∎

Example 1.10. For a set S, let $R\{S\} \cong \bigoplus_S R$ be the free R-module on the set S. Then the R-linear dual of $R\{S\}$ is

$$\mathrm{Hom}_R(R\{S\}, R) \cong \mathrm{Hom}_{\mathbf{Set}}(S,R) = R^S.$$

Moreover, $R\{S\} \otimes_R R\{T\} \cong R\{S \times T\}$ (see MA654 homework). Therefore Proposition 1.2 follows from Lemma 1.9.

Proposition 1.11. *Let A be an R-algebra that is free of finite rank as an R-module. Then $C := \mathrm{Hom}_R(A,R)$ is an R-coalgebra.*

Proof. The counit $\varepsilon : C \longrightarrow R$ is simply the dual of the unit $u : R \longrightarrow A$. Dualizing the multiplication gives an R-module map

$$\Delta = \mu^* : C = \mathrm{Hom}_R(A,R) \longrightarrow \mathrm{Hom}_R(A \otimes A, R).$$

So if A is free of finite rank, then the comultiplication on the dual C is defined as

$$C = \mathrm{Hom}_R(A,R) \xrightarrow{\mu^*} \mathrm{Hom}_R(A \otimes A, R) \cong \mathrm{Hom}_R(A,R) \otimes \mathrm{Hom}_R(A,R) = C \otimes C.$$

∎

More generally, if A is as in Proposition 1.11 and $\{a_i\}$ is a basis for A, then Δ is given by

$$\Delta(\lambda) = \sum_{i,i'} \lambda(a_i \cdot a_{i'}) a_i^* \otimes a_{i'}^*.$$

In particular, on the basis element $\lambda = a_j^*$ for $C = A^*$, this gives

$$\Delta(a_j^*) = \sum_{i,i'} n_{i,i'}^j \, a_i^* \otimes a_{i'}^*,$$

where $a_i \cdot a_{i'} = \sum_j n_{i,i'}^j a_j$.

Example 1.12. Consider $A = R[x]/(x^2 = 1)$. Then A is free of rank two as an R-module with basis $\{1, x\}$. The dual $C = \mathrm{Hom}_R(A,R)$ is also a free R-module, with basis 1^* and x^*. The counit ε is the projection onto the summand 1^*, and the comultiplication $\Delta : C \longrightarrow C \otimes C$ is

$$1^* \mapsto 1^* \otimes 1^* + x^* \otimes x^*, \qquad x^* \mapsto 1^* \otimes x^* + x^* \otimes 1^*.$$

This coalgebra is isomorphic to R^{C_2}, where C_2 is the cyclic group of order two.

Fri, Jan. 20

Exercise 1.13. Repeat the above example, starting from

$$A = R[x]/(x^3 = 1) \qquad \text{or} \qquad A = R[x]/(x^n = 1).$$

Exercise 1.14. Repeat the above example, starting from
$$A = R[x,y]/(x^2 = 1, y^2 = 1) \qquad \text{or} \qquad A = R[x,y]/(x^2 = 1, y^3 = 1).$$

Exercise 1.15. Repeat the above, starting from
$$A = R\langle x,y\rangle/(x^2 = 1, y^3 = 1, xy = y^2x).$$

1.2. **Bialgebras.** Recall that we began by considering R^M and noticing that this has a coalgebra structure. But this was after noticing that R^S is *always* a commutative ring. Moreover, by functoriality, we saw that both $\varepsilon : R^M \longrightarrow R$ and $R^M \xrightarrow{\Delta} R^M \otimes R^M$ are ring homomorphisms.

Definition 1.16. An R-bialgebra B is an R-module equipped with algebra and coalgebra structures, such that
$$\varepsilon : B \longrightarrow R$$
and
$$\Delta : B \longrightarrow B \otimes B$$
are R-algebra maps. We say that B is commutative or cocommutative if μ or Δ, respectively, are commutative.

Remark 1.17. Recall that $B \otimes B$ is again an R-algebra, with unit $1 \otimes 1$ and multiplication defined by $(a \otimes b) \cdot (c \otimes d) := (ac \otimes bd)$.

Thus R^M is a commutative bialgebra.

Proposition 1.18. *If B is R-bialgebra that is free of finite rank as an R-module, then the dual $B^* = \mathrm{Hom}_R(B, R)$ inherits an R-bialgebra structure as well.*

Example 1.19. If M is a monoid, the **monoid ring** is
$$R[M] = \oplus_M R.$$
The unit $e \in M$ serves as the algebra unit, and the multiplication is induced by the multiplication in M. Note that the R-linear dual is
$$\mathrm{Hom}_R(R[M], R) \cong \mathrm{Hom}_{\mathbf{Set}}(M, R) = R^M.$$
Assuming that M is finite, we conclude that $R[M]$ can be identified with the dual of R^M, so that it inherits a bialgebra structure. The algebra structure is the one we introduced above. The coalgebra structure comes from the algebra structure on R^M. Since the unit of R^M is the constant function at 1, which is the sum $1 = \sum_m m^*$, it follows that the counit of $R[M]$ is given by
$$R[M] \xrightarrow{\varepsilon} R$$
$$\varepsilon(m) = 1$$
for all $m \in M$. Given two elements n, k in M, we have that
$$n^* \cdot k^* = \begin{cases} n^* & n = k \\ 0 & \text{else} \end{cases}$$
in R^M. It follows that the coproduct on $R[M]$ is
$$R[M] \xrightarrow{\Delta} R[M] \otimes_R R[M]$$
$$\Delta(m) = m \otimes m.$$

Note that whereas R^M was commutative, the monoid ring $R[M]$ is cocommutative.

Exercise 1.20. Check that this makes $R[M]$ into a bialgebra.

Example 1.21. The polynomial ring $R[x]$ can be thought of as the monoid ring $R[\mathbb{Z}_{\geq 0,+}]$, which gives it a bialgebra structure. The coalgebra structure maps are

$$\varepsilon(x^n) = 1, \quad \forall n, \qquad \Delta(x^n) = x^n \otimes x^n, \quad \forall n.$$

Mon, Jan. 23

There are several ways to state the bialgebra conditions.

Proposition 1.22. *Suppose that B is an R-module equipped with algebra and coalgebra structures. The following are equivalent*

(1) $\varepsilon : B \longrightarrow R$ *and* $\Delta : B \longrightarrow B \otimes_R B$ *are R-algebra maps.*
(2) $\eta : R \longrightarrow B$ *and* $\mu : B \otimes_R B \longrightarrow B$ *are R-coalgebra maps.*
(3) *The following diagrams commute:*

$$\begin{tikzcd} R \arrow[rr, equal] \arrow[dr, "\eta"'] & & R \\ & B \arrow[ur, "\varepsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} B \otimes_R B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"] & R \otimes_R R \arrow[d, "\mu", "\cong"'] \\ B \arrow[r, "\varepsilon"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta"] \arrow[d, "\Delta"', "\cong"] & B \arrow[d, "\Delta"] \\ R \otimes_R R \arrow[r, "\eta\otimes\eta"'] & B \otimes_R B \end{tikzcd}$$

$$\begin{tikzcd} B \otimes_R B \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"'] & B \otimes_R B \otimes_R B \otimes_R B \arrow[r, "1\otimes twist\otimes 1"] & B \otimes_R B \otimes_R B \otimes_R B \arrow[d, "\mu\otimes\mu"] \\ B \arrow[rr, "\Delta"'] & & B \otimes_R B \end{tikzcd}$$

1.3. **Graded (bi)algebras.** Many of the examples we will be interested in also coming with a grading:

Definition 1.23. A **graded R-module** M_* is a tuple $\{M_i\}$ of R-modules. We will usually index graded modules on the set $\mathbb{Z}_{\geq 0}$. A map $f_* : M_* \longrightarrow N_*$ of graded modules is a tuple of module maps $f_i : M_i \longrightarrow N_i$.

Example 1.24. We will often consider the R-module R as a graded module concentrated in degree zero (meaning that the kth graded piece is trivial when $k \neq 0$).

Given a graded module M_*, we often consider the direct sum $\bigoplus_i M_i$. Conversely, decomposing an R-module M into a direct sum $M \cong \bigoplus_i M_i$ imparts M with a grading.

Definition 1.25. (1) A **graded R-algebra** A is simultaneously a graded R-module and an R-algebra, such that the structure maps $\mathbb{R} \xrightarrow{\eta} A$ and $A \otimes_R A \xrightarrow{\mu} A$ are graded maps.
(2) A **graded R-coalgebra** C is simultaneously a graded R-module and an R-coalgebra in the same way.
(3) A **graded R-bialgebra** B is simultaneously a graded R-module and an R-bialgebra in the same way.

Remark 1.26. In a graded context, (co)commutativity will always be interpreted in the graded sense, meaning that $a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a$.

Example 1.27. The grading on the polynomial ring $R[x]$ is determined once we declare the degree of the generator x. If we wish $R[x]$ to be graded-commutative, this restricts our possible choices for $\deg(x)$. In particular, if we wish to place x in *odd* degree, then graded-commutativity gives

$$x \cdot x = (-1)^{\deg(x)\deg(x)} x \cdot x = -x \cdot x,$$

so that $2x^2 = 0$. Thus we can only place x in odd degree if $2 = 0$ in R. Otherwise, we are forced to place x in even degree.

To further equip $R[x]$ with a graded bialgebra structure, we are forced, by degree considerations, to define $\varepsilon(x^n) = 0$ if $n > 0$. Similarly, since the graded module $R[x]$ is trivial in degrees between zero and $\deg(x)$, this forces the choice $\Delta(x) = 1 \otimes x + x \otimes 1$. We are then forced to set

$$\Delta(x^n) = (1 \otimes x + x \otimes 1)^n = \sum_k \binom{n}{k} x^k \otimes x^{n-k}.$$

Remark 1.28. An element $x \in C$ of a bialgebra is said to be **primitive** if it satisfies $\Delta(x) = 1 \otimes x + x \otimes 1$.

Example 1.29. We similarly make $R[x_1, x_2, \dots, x_n]$ into a bialgebra by making the x_i's primitive.

Wed, Jan. 25

Last time, we introduced graded (bi/co)-algebras. Signs can pop up unexpectedly here. For instance, recall that in a bialgebra, the comultiplication $\Delta : B \longrightarrow B \otimes B$ is an algebra map. Beware that in the graded setting, the multiplication on $B \otimes B$ is given by

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{bc} ac \otimes bd.$$

So this sign can appear in comparing $\Delta(xy)$ with $\Delta(x)\Delta(y)$.

Example 1.30. (Suppose that R is a field.) We saw before that $\mathrm{H}^0(X; R) \cong R^{\pi_0(X)}$. So if X is a topological monoid, so that $\pi_0(X)$ becomes a monoid, then we get a bialgebra structure on $\mathrm{H}^0(X; R)$. Similarly, at least if R is a field, we get a bialgebra structure on $\mathrm{H}_0(X; R) \cong R[\pi_0(X)]$.

Moreover, the graded ring $\mathrm{H}^*(X; R)$ becomes a graded bialgebra if X is a topological monoid. The coproduct is

$$\mathrm{H}^*(X; R) \xrightarrow{\mu^*} \mathrm{H}^*(X \times X; R) \cong \mathrm{H}^*(X; R) \otimes_R \mathrm{H}^*(X; R).$$

The counit is $\mathrm{H}^*(X; R) \xrightarrow{e^*} \mathrm{H}^*(\{e\}; R) \cong R$.

Remark 1.31. We say that a graded algebra B is **connected** if B is concentrated in nonnegative degrees, and moreover $B_0 = R$. A prototypical example is $\mathrm{H}^*(X; R)$, where X is path-connected.

Example 1.32. Take $X = S^1$. Then $\mathrm{H}^*(S^1; R) \cong R[x]/x^2 = 0$, where $\deg(x) = 1$. This is a truncated polynomial algebra. The counit sends x to 0, and x is primitive. To see the latter claim, we wish to understand

$$\mu^* : \mathrm{H}^1(S^1; R) \longrightarrow \mathrm{H}^1(S^1 \times S^1; R) \cong \Big(\mathrm{H}^1(S^1; R) \otimes_R \mathrm{H}^0(S^1; R)\Big) \oplus \Big(\mathrm{H}^0(S^1; R) \otimes_R \mathrm{H}^1(S^1; R)\Big).$$

But the counit law tells us that the term on the left must be $x \otimes 1$, and similarly the other counit law tells us that the term on the right must be $1 \otimes x$. Note that in order for Δ to be an *algebra* map, we must have $\Delta(x)^2 = 0$ since $x^2 = 0$. Now

$$\Delta(x)^2 = (x \otimes 1 + 1 \otimes x)(x \otimes 1 + 1 \otimes x) = x^2 \otimes 1 + x \otimes x - x \otimes x + 1 \otimes x^2 = 0.$$

Here, the sign saves the day to make the terms cancel!

We will often refer to the algebra $R[x]/x^2$ as an **exterior algebra** on x and denote it by $E_R(x)$.

Example 1.33. Recall ([H], §3D) that $\mathbb{RP}^3 \cong SO(3)$. It follows that

$$\mathrm{H}^*(\mathbb{RP}^3; \mathbb{F}_2) \cong \mathbb{F}_2[x]/x^4$$

inherits a biaglebra structure. As above, the grading forces the counit to be given by $\varepsilon(x) = 0$. Similarly, x must be primitive for degree reasons. The bialgebra properties then give

$$\Delta(x^2) = x^2 \otimes 1 + 1 \otimes x^2, \qquad \Delta(x^3) = x^3 \otimes 1 + x^2 \otimes x + x \otimes x^2 + 1 \otimes x^3.$$

Note also that $0 = \Delta(x^4) = \Delta(x)^4 = 0$, for consistency.

Remark 1.34. Given the last two examples, it is natural to wonder about $\mathbb{F}_2[x]/x^3$. But if you try to make it into a bialgebra in the same way, you run into trouble because

$$\Delta(x^3) = \Delta(x)^3 = x^2 \otimes x + x \otimes x^2 \neq 0.$$

1.4. **Classification theorems.**

Definition 1.35. We say that a bialgebra B is **primitively generated** if it is generated as an algebra by the space of primitive elements.

Proposition 1.36. *Suppose that A^* is a finite-dimensional, connected, commutative bialgebra over a field k of characteristic 0. Suppose moreover that A^* is primitively generated. Then*

$$A^* \cong \bigotimes E(x_{\mathrm{odd}})$$

is an exterior algebra on odd-degree generators.

Proof. Step one: we show that the space $P \subseteq A^*$ of primitive elements is concentrated in odd degrees. Suppose $x \in P$ lies in degree $2k$. Take N large enough so that $x^N = 0$. Then

$$0 = \Delta(x^N) = x^N \otimes 1 + 1 \otimes x^N + \sum_i \binom{N}{i} x^i \otimes x^{N-i}.$$

All terms live in different graded pieces of $A \otimes A$ and must be zero. In particular, we have $0 = Nx \otimes x^{N-1}$. Since our field is characteristic zero, we conclude that $x^{N-1} = 0$. Repeating as necessary, we conclude that $x = 0$.

Fri, Jan. 27

Step two: We already know that A^* is generated in odd degrees. By graded-commutativity, these odd-degree generators must all square to zero (since we are in characteristic zero). It thus only remains to show that there are no additional relations between these generators. In other words, supposing that $\{x_1, \dots, x_n\}$ is an independent set of primitive generators, we wish to show that the set of products in the x_i's is independent.

We argue by induction on n. In the base case $n = 1$, there is nothing to show. By assumption, x_n is independent of $\{x_1, \dots, x_{n-1}\}$. It follows that $x_n \notin E(x_1, \dots, x_{n-1})$ since x_n is primitive, whereas

Lemma 1.37. *In the bialgebra $E(x_1, \dots, x_{n-1})$, in which each x_i is primitive, we have*

$$P\Big(E(x_1, \dots, x_{n-1})\Big) = \mathrm{Span}\{x_1, \dots, x_{n-1}\}.$$

(The LHS is the vector space of primitive elements.)

Now suppose we have a relation

$$p + q \cdot x_n = 0,$$

where p and q are polynomial in $x_1, \dots, x_{n-1}$. Applying the coproduct gives

$$\Delta(p) + \Delta(q)(x_n \otimes 1 + 1 \otimes x_n) = 0.$$

The expression $q \otimes x_n$ appears as one of the terms in this expression. By the above, this term is independent of the others, as the other terms live in $A \otimes E(x_1, \dots, x_{n-1})$. It follows that the term $q \otimes x_n$ must be zero, so we conclude that $q = 0$, which implies $p = 0$. ■

In fact, the primitively generated hypothesis is not necessary.

Theorem 1.38. *[Hopf] Suppose that A^* is a finite-dimensional, connected, commutative bialgebra over a field k of characteristic 0. Then*
$$A^* \cong \bigotimes E(x_{\text{odd}})$$
is an exterior algebra on odd-degree generators.

Proof. Consider $I = \ker(\varepsilon) \subseteq A$. We will use this notation often for this **augmentation ideal**. Since A is connected, I is just $A^{>0}$. We can consider the various powers of the ideal I, which give a **filtration** of our graded algebra A^*:
$$A^* = I^0 \supset I \supset I^2 \supset I^3 \supset \dots.$$
We can then consider the various quotients I^n/I^{n+1}, which are graded k-vector spaces. These assemble together to form the **associated graded algebra** to A^* with respect to the ideal I:
$$\mathrm{gr}_I^* A^* = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$
The algebra structure is given by maps
$$I^n/I^{n+1} \otimes I^k/I^{k+1} \longrightarrow I^{n+k}/I^{n+k+1}.$$
Notice that each I^n/I^{n+1} is already graded, so that $\mathrm{gr}_I^* A$ becomes a **bigraded ring**. We refer to the grading inherited from A^* as the **internal grading** and the new grading in the direct sum decomposition as the **filtration grading**. With respect to the internal grading, $\mathrm{gr}_I^* A^*$ is a graded-commutative, connected finite-dimensional algebra. It is generated by the subspace I/I^2.

Mon, Jan. 30

We wish to apply Proposition 1.36 to $\mathrm{gr}_I^* A^*$. To do so, we must transport the coalgebra structure from A^* to $\mathrm{gr}_I^* A^*$. In A^*, the coproduct on I takes shape
$$I \xrightarrow{\Delta} (I \otimes A) \oplus (A \otimes I).$$
It follows that, on I^n, it is
$$I^n \xrightarrow{\Delta} \bigoplus_{k=0}^{n} I^k \otimes I^{n-k}.$$
By considering where this sends the subspace I^{n+1}, you can see that this descends to the quotient
$$I^n/I^{n+1} \xrightarrow{\Delta} \bigoplus_{k=0}^{n} I^k/I^{k+1} \otimes I^{n-k}/I^{n-k+1}.$$
Since the multiplication and comultiplication are both induced from A^*, it follows that $\mathrm{gr}_I^* A^*$ is also a bialgebra. Moreover, the coproduct on the generating set I/I^2 takes the form
$$I/I^2 \xrightarrow{\Delta} (I/I^2 \otimes A/I) \oplus (A/I \otimes I/I^2),$$
which forces the generators to be primitive. We may now apply Proposition 1.36 to conclude that $\mathrm{gr}_i^* A^*$ is an exterior algebra on the generating set I/I^2.

Now if we lift the generators of $\mathrm{gr}_I^* A^*$ to generators of A^*, it follows that they generate an exterior algebra, since any relation would descend to define a relation in $\mathrm{gr}_I^* A^*$. ■

There are similar results in positive characteristic. For example

Theorem 1.39. [C, Thm 2.5.C] *Suppose that k is a field of characteristic 2 such that every element has a square root (we say k is 'perfect'). If A^* is as in Theorem 1.38, then A is of the form*

$$A^* \cong \bigotimes_i k[x_i]/(x^{2^{n_i}}),$$

where the x_i are primitive and may be of any (positive) degree.

We have a similar classification theorem in the ungraded case.

Theorem 1.40. *Suppose that k is algebraically closed of characteristic zero and that B is a commutative, finite bialgebra. Then B is isomorphic to the bialgebra k^M for some monoid M.*

Remark 1.41. If we do not assume k to be algebraically closed, then the statement is not even true as *algebras*. For example, $\mathbb{C}$ is a finite $\mathbb{R}$-algebra, but it is not isomorphic to $\mathbb{R}^S$ for some set S.

Note that this is different from what we saw in the graded situation: all positive degree elements in an A^* as in Theorem 1.38 are nilpotent, whereas a ring of functions R^M will not have any nilpotent elements (since R has none).

Corollary 1.42. *Let $k = \bar{k}$ of characteristic zero, and let B be a cocommutative, finite R-bialgebra. Then $B \cong R[M]$ for some monoid M.*

Proof of Theorem 1.40. Let $M = \mathrm{mSpec}(B)$, the maximal ideal spectrum of B. We give this a monoid structure as follows: We take $I = \ker \varepsilon$ as the unit element of M. The multiplication is

$$M \times M = \mathrm{mSpec}(B) \times \mathrm{mSpec}(B) \cong \mathrm{mSpec}(B \otimes B) \xrightarrow{\Delta^*} \mathrm{mSpec}(B) = M.$$

Hilbert's Nullstellensatz then allows us to identify B with the ring of functions on M, its maximal ideal spectrum, assuming that B is reduced (has no nilpotent elements). But in fact a Theorem of Cartier says that commutative bialgebras in characteristic zero are reduced! ∎

Wed, Feb. 1

2. Hopf algebras

Recall that bialgebras were modeled on monoid rings $R[M]$ or functions on a monoid R^M. If we ask in addition the monoid to be a group, we arrive at the notion of a Hopf algebra.

Definition 2.1. A Hopf algebra H is a bialgebra equipped with a linear map (called the **antipode** or **conjugation**) $\chi : H \xrightarrow{S} H$ such that the diagrams

$$\begin{tikzcd} H \otimes H \arrow[rr, "\mathrm{id}\otimes\chi"] & & H \otimes H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & R \arrow[r, "\eta"'] & H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[rr, "\chi\otimes\mathrm{id}"] & & H \otimes H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & R \arrow[r, "\eta"'] & H \end{tikzcd}$$

These diagrams are precisely what you get by translating the defining property of group inverses ($g \cdot g^{-1} = e = g^{-1} \cdot g$) into diagrams. Just as we always have $e^{-1} = e$ in a group, we similarly get that $\chi(1) = 1$ in a Hopf algebra.

Example 2.2. The group ring $R[G]$ is a Hopf algebra for any group G. The antipode is specified on the basis elements by $\chi(g) = g^{-1}$.

Example 2.3. Similarly, the dual bialgebra R^G is also a Hopf algebra. The antipode is given by $\chi(g^*) = (g^{-1})^*$.

In the graded setting, we ask further that χ be a graded linear map.

Example 2.4. Let $E_R(x)$ be a bialgebra (so x is in odd degree if the characteristic of R is not 2). Then the antipode $\chi(x) = -x$ makes $E_R(x)$ into a Hopf algebra.

Example 2.5. Let $R[x]$ be a bialgebra (so x is in even degree if the characteristic of R is not 2). Then the antipode $\chi(x^n) = (-1)^n x^n$ makes $R[x]$ into a Hopf algebra.

Example 2.6. If we dualize the previous example, write x_n for the dual basis element to x^n. Then $\Gamma(x) := \mathrm{Hom}(R[x], R) \cong \bigoplus_n R\{x_n\}$. The unit is the identification $R \cong R\{x_0\}$, and the counit projects onto the summand $R\{x_0\}$. The comultiplication is

$$R\{x_n\} \longrightarrow \bigoplus_i R\{x_i\} \otimes R\{x_{n-i}\}.$$

What is perhaps surprising is the product formula, which comes from the coproduct in $R[x]$. Since the term $x^i \otimes x^j$ appears with coefficient $\binom{i+j}{i}$ in the expression for $\Delta(x^{i+j})$, it follows that, in $\Gamma(x)$, we have $x_i \cdot x_j = \binom{i+j}{i} x_{i+j}$. This is known as the **divided powers algebra**. The antipode is $\chi(x_n) = (-1)^n x_n$ just as in the polynomial algebra.

Fri, Feb. 3

Proposition 2.7. *Suppose that B^* is a connected, graded bialgebra. Then B^* admits a unique antipode.*

Proof. We argue by induction on degree. If $x \in B^1$, we know (since B^* is connected) that x is primitive, and $\varepsilon(x) = 0$. Thus

$$\mu(\mathrm{id} \otimes \chi)\Delta(x) = \mu(\mathrm{id} \otimes \chi)(x \otimes 1 + 1 \otimes x) = x + \chi(x)$$

must be zero, which forces $\chi(x) = -x$. Now suppose that we have defined χ in degrees $< n$. If $x \in B^n$, we have $\varepsilon(x) = 0$, so that

$$\mu(\mathrm{id} \otimes \chi)\Delta(x) = \mu(\mathrm{id} \otimes \chi)(x \otimes 1 + 1 \otimes x + \sum_{(x)} (x_1 \otimes x_2) = x + \chi(x) + \sum_{(x)} x_i \chi(x_2)$$

must be zero. Since each x_2 is in degree less than n, we have already defined $\chi(x_2)$, so that this gives an inductive formula for $\chi(x)$. ∎

Notation 2.8. In the proof, we employed **Sweedler's notation** for the coproduct:

$$\Delta(x) = \sum_{(x)} x_1 \otimes x_2.$$

Proposition 2.9. *Suppose H is a Hopf algebra. Then $H \xrightarrow{\chi} H$ is an algebra anti-homomorphism, meaning that $\chi(1) = 1$ and $\chi(xy) = (-1)^{\deg(x)\deg(y)} \chi(y)\chi(x)$.*

In order to prove this, it is useful to introduce the following idea. Suppose that C is a coalgebra and A is an algebra. Then there is a way to combine two maps $C \rightrightarrows A$: define the **convolution** operation by

$$f * g : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.$$

In Sweedler notation, this is written as $(f * g)(x) = \sum_{(x)} f(x_1) g(x_2)$.

Exercise 2.10. If C is a coalgebra and A is an algebra, then the convolution makes $\mathrm{Hom}(C, A)$ into an R-algebra with unit given by $C \xrightarrow{\varepsilon} R \xrightarrow{\eta} A$. (*This will be on homework.*)

In this language, if H is a Hopf algebra, then the antipode χ is the multiplicative inverse to id_H in the convolution algebra $\mathrm{Hom}(H, H)$.

Proof of Proposition 2.9. Recall that we can consider $H \otimes H$ as a coalgebra, with comultiplication

$$H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} H \otimes H \otimes H \otimes H$$

and counit $\varepsilon \otimes \varepsilon$. In Sweedler notation, the comultiplication is $a \otimes b \mapsto \sum_{(a),(b)} (-1)^{\deg(b_1)\deg(a_2)} a_1 \otimes b_1 \otimes a_2 \otimes b_2$

In $\mathrm{Hom}(H \otimes H, H)$, we want to compare $\chi \circ \mu$ with $\mu \circ \tau \circ (\chi \otimes \chi)$. We will show that they are both multiplicative inverses to μ.

$$\begin{aligned} \chi\mu * \mu &:= \mu \circ (\chi\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta) \\ &= \mu \circ (\chi \otimes \mathrm{id}) \circ (\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta) \end{aligned}$$

Since μ is a coalgebra map (from the definition of bialgebra), we can rewrite this as

$$\mu \circ (\chi \otimes \mathrm{id}) \circ \Delta \circ \mu = \varepsilon\mu = \varepsilon \otimes \varepsilon,$$

where the last equality holds since ε is an algebra map. For variety, we analyze the convolution $\mu * (\mu\tau(\chi \otimes \chi))$ using Sweedler notation:

$$\begin{aligned} \Big(\mu * (\mu\tau(\chi \otimes \chi))\Big)(a \otimes b) &= \sum_{(a),(b)} (-1)^{\deg(a_2)\deg(b)} a_1 b_1 \chi(b_2)\chi(a_2) \\ &= \sum_{(a)} a_1 \varepsilon(b) a_2 = \sum_{(a)} a_1 a_2 \varepsilon(b) = \varepsilon(a)\varepsilon(b) = (\varepsilon \otimes \varepsilon)(a \otimes b). \end{aligned}$$

Note that the sign disappeared since $\varepsilon(b)$ is only nonzero if $\deg(b) = 0$. Now the usual trick showing that a left inverse and right inverse agree finish the proof:

$$\chi\mu = \chi\mu * \Big(\mu * (\mu\tau(\chi \otimes \chi))\Big) = \Big(\chi\mu * \mu\Big) * (\mu\tau(\chi \otimes \chi))) = \mu\tau(\chi \otimes \chi)$$

■

Mon, Feb. 6

Proposition 2.11. *Suppose that H is a commutative (or cocommutative) Hopf algebra. Then $\chi^2 = \mathrm{id}$.*

Exercise 2.12. This will be on your next homework.

Example 2.13. If H and H' are Hopf algebras, then $H \otimes H'$ inherits a Hopf algebra structure as well. In particular, we get a Hopf algebra structure on $R[x,y,z] \cong R[x] \otimes R[y] \otimes R[z]$.

Example 2.14. Consider the noncommutative ring $R\langle x,y\rangle$. We similarly make this into a Hopf algebra by making both x and y primitive. Since we are not hoping for commutativity, there is no harm in placing the generators in degree 1 here. We again get $\chi(x) = -x$ and $\chi(y) = -y$. But note that Proposition 2.9 gives that $\chi(xy) = -yx$ if we place x and y in degree 1. More generally, we get

$$\chi(x^n y^k) = (-1)^{nk+n+k} y^k x^n.$$

This is also known as the **tensor algebra** $R\langle x,y\rangle \cong T(x,y)$.

Example 2.15. More generally, if $V \cong k\{x_1, \dots, x_n\}$ is an n-dimensional vector space, the **tensor algebra** $T(V)$ is the noncommutative ring $k\langle x_1, \dots, x_n\rangle$. We place each x_i in degree 1, and there is a unique Hopf algebra structure. Necessarily, each x_i is primitive, and the coproduct can be described as follows. Given natural numbers p and q, a (p,q)-shuffle is a permutation σ of $n = p+q$ letters such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p), \qquad \sigma(p+1) < \cdots < \sigma(n).$$

Think of shuffle a set of p cards with a set of q cards. We denote by $Ш_{p,q}$ the set of (p,q)-shuffles. Then the coproduct formula in $T(V)$ is

$$\Delta(v_1 \cdots v_k) = \sum_{\substack{p+q=k \\ \sigma \in Ш_{p,q}}} (-1)^{\mathrm{sgn}(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(k)}.$$

Exercise 2.16. The tensor algebra $T(V)$ is cocommutative.

Before giving more examples, we return to discuss classification theorems in the non-finite case.

Notation 2.17. Let A be a connected bialgebra and $I \subseteq A$ the augmentation ideal. The R-module $QA := I/I^2$ is called the **module of indecomposables** of A. This is the submodule spanned by a set of algebra generators.

Theorem 2.18 (Leray)**.** [C, Thm 3.8.3] *Let $R = k$ be a field of characteristic zero. Let H be a connected, commutative Hopf algebra over k. Then H is isomorphic, as an algebra, to the free, graded commutative algebra on QH.*

Decomposing QH into its even degree part QH^+ and its odd degree part QH^-, the previous result says that $H \cong P(QH^+) \otimes E(QH^-)$. Just as in the finite case, we have a similar result in positive characteristic.

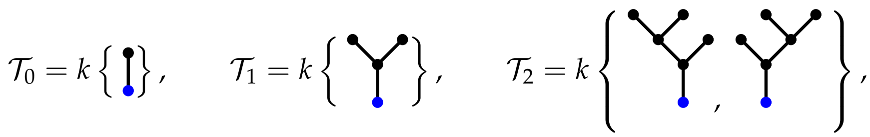
Theorem 2.19 (Borel)**.** [C, Thm 2.5.B] *Let $R = k$ be a perfect field of characteristic p. Let H be a connected, commutative Hopf algebra over k. Then*

$$A \cong \bigotimes_i k[x_i] \otimes \bigotimes_j k[y_j]/y_j^{p^{n_i}}$$

as algebras.

2.1. **Rooted trees and quasi-symmetric functions.**

Example 2.20. (The Hopf algebra of rooted trees, [LR]) Let $\mathcal{T}_n$ be the k-vector space with basis given by the set of rooted, binary, planar trees with n internal vertices (and thus $n+1$ 'leaves'). The root is a choice of distinguished external vertex. A tree is binary when each internal vertex is trivalent. Here 'planar' refers to having a chosen embedding in the plane, so that we know which branch is "left" and which branch is "right" at a vertex.



The dimension of $\mathcal{T}_n$ is the **Catalan number** $\frac{(2n)!}{(n+1)!n!}$. We make $\mathcal{T} := \oplus_n \mathcal{T}_n$ into a graded Hopf algebra as follows.

First, we declare the tree  to be the multiplicative unit. Now suppose that T_1 and T_2 are trees in positive degrees. There is an operation on trees called **grafting** which identifies the roots of the two trees, and attaches a new root to the result. The grafting operation is sometimes denoted by the symbol $\vee$. We may write

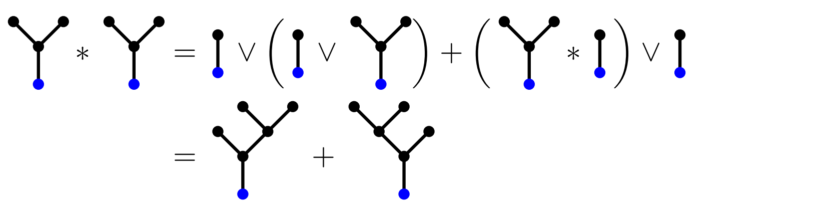
$$T_1 = L_1 \vee R_1, \qquad T_2 = L_2 \vee R_2.$$

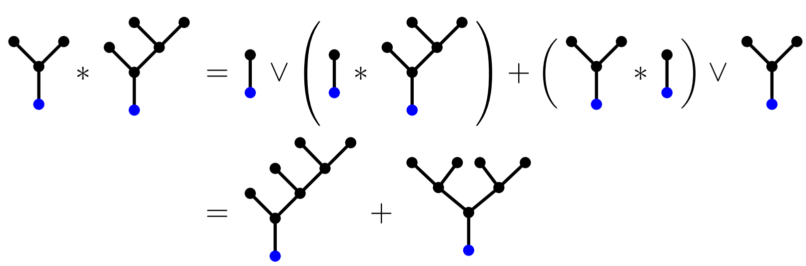
We then (inductively) define the multiplication in $\mathcal{T}$ by

$$T_1 * T_2 := L_1 \vee (R_1 * T_2) + (T_1 * L_2) \vee R_2.$$

Wed, Feb. 8

Some examples are

(2.21)


(2.22)


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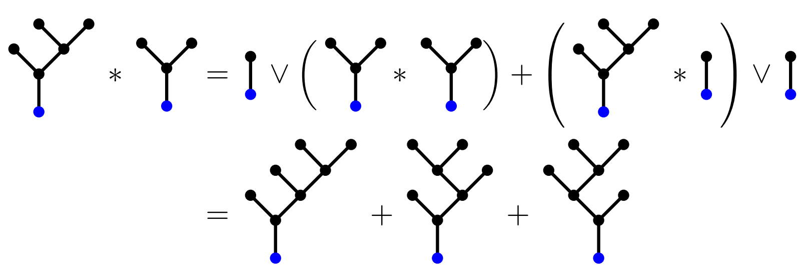


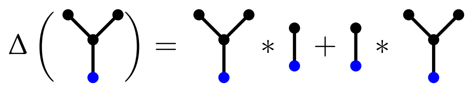
In particular, the last two examples show this is not commutative.

The coproduct is similarly given by a recursive formula. If $T = L \vee R$, then

$$\Delta(L \vee R) = \sum_{(L)(R)} (L_1 * R_1) \otimes (L_2 \vee R_2) + (L \vee R) \otimes$$  .

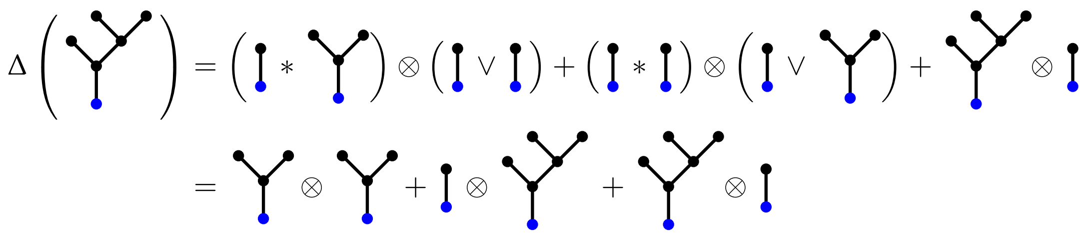
For example, we have

(2.24)

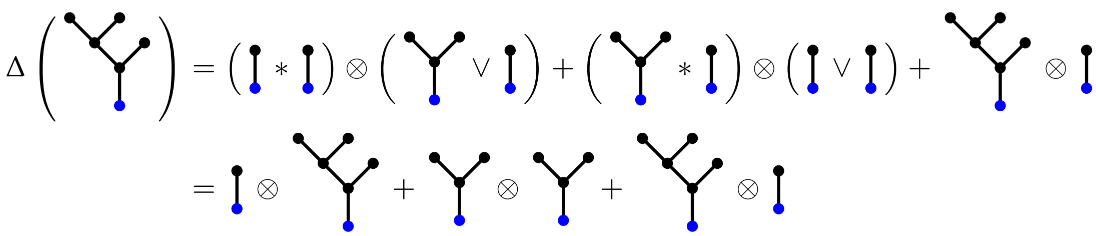


as must be the case since $\mathcal{T}$ is connected.

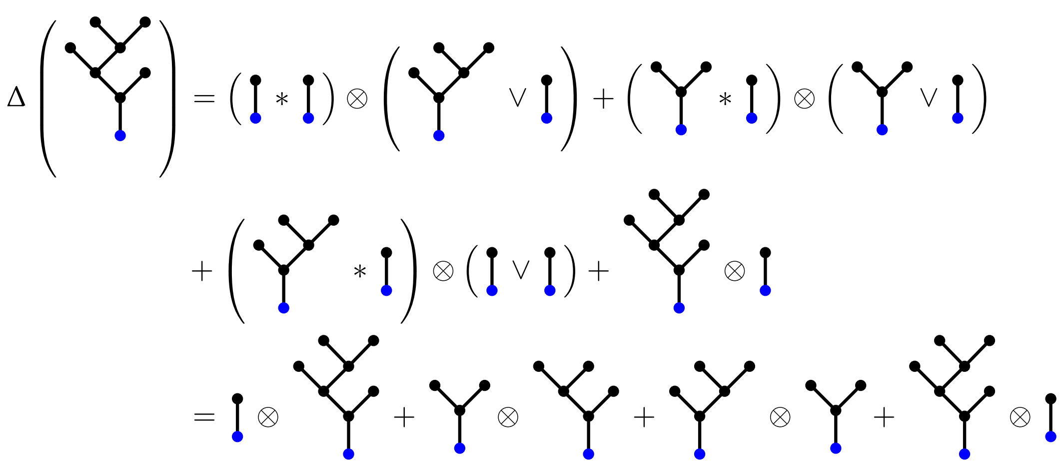
(2.25)



(2.26)



(2.27)

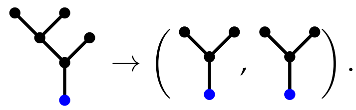


The last example shows that $\mathcal{T}$ is not cocommutative.

Theorem 2.28 (Loday-Ronco, [LR, Thm 3.8])**.** *The Hopf algebra $\mathcal{T}$ is isomorphic to a tensor algebra $T(\mathcal{W})$ on the set*

$$\mathcal{W} = \Big\{$$  $$\vee T \mid T \in \mathcal{T} \Big\}.$$

Example 2.29. (The dual to $\mathcal{T}$, [AS]) It turns out the dual Hopf algebra $\mathcal{T}\mathrm{Sym} := \mathcal{T}^\vee$ has a somewhat nicer structure. We denote by $\{F_t \mid t \in \mathcal{T}\}$ the dual basis. Say a tree t **divides** at a leaf ℓ to trees t_1 and t_2 if t_1 consists of the tree to the left of (and including) ℓ, whereas t_2 is the tree to the right of ℓ. We write $t \to (t_1, t_2)$, suppressing the leaf ℓ. For example, dividing at the middle leaf gives

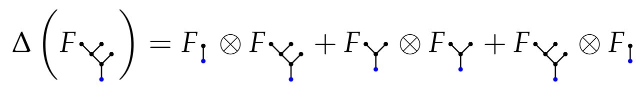


Then the coproduct in $\mathcal{T}$Sym is given by

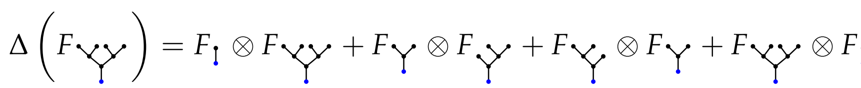
$$\Delta(F_t) = \sum_{t \to (t_1, t_2)} F_{t_1} \otimes F_{t_2}.$$

Fri, Feb. 10

Some examples of the coproduct calculation are



and

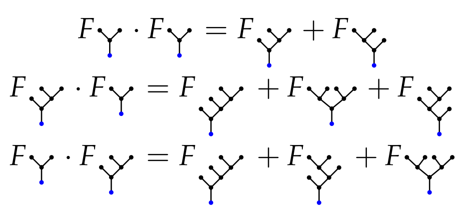


The second example shows that the $\mathcal{T}$Sym is not cocommutative.

For the product, suppose that $t \in \mathcal{T}_n$ and $s \in \mathcal{T}_k$. Let $(t_0, \ldots, t_k)$ be the result of subdividing t at a multiset (allows repetition) of leaves. Define the tree $(t_0, \ldots, t_k)/s$ to be the result of planting each tree t_i in leaf i of s. Then the product in $\mathcal{T}$Sym is given by

$$F_t \cdot F_s := \sum_{t \to (t_0, \ldots, t_k)} F_{(t_0, \ldots, t_k)/s}.$$

For example, we have



The last two examples show that $\mathcal{T}$Sym is not commutative. Just as $\mathcal{T}$ was neither commtuative nor cocommutative, the same is true of $\mathcal{T}$Sym.

Mon, Feb. 13

See Theorem 6.1 of [AS] for an explicit description of the antipode in $\mathcal{T}$Sym (under a different choice of basis).

Example 2.30. (Quasi-symmetric functions) Recall that a **composition** of a natural number n is an (ordered) tuple $(a_1, \dots, a_k)$ of positive integers that sum to n. The algebra $\mathcal{Q}$Sym of **quasi-symmetric functions** is a graded, connected Hopf algebra. In degree n, it has the **monomial** basis $\{M_\alpha \mid \alpha \text{ is a composition of } n\}$. Note that there is a single composition of 0, namely the empty composition, making $\mathcal{Q}$Sym connected. One interpretation is as the power series

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{a_1} \dots x_{i_k}^{a_k}.$$

in the ring $R[\![x_1, x_2, \dots]\!]$. Note that the power series M_α is not symmetric in the x_i, but it is shift-invariant, which explains the name "quasi-symmetric". This interpretation as power series gives the multiplication. This can be described in terms of **quasi-shuffles**: a quasi-shuffle of the compositions $(a_1, \dots, a_k)$ and $(b_1, \dots, b_n)$ is the result of first shuffling, and next replacing some adjacent pairs (a_i, b_j) by $a_i + b_j$. Denoting by $q\text{Ш}(\alpha, \beta)$ the set of quasi-shuffles of α and β, the multiplication is

$$M_\alpha \cdot M_\beta = \sum_{\gamma \in q\text{Ш}(\alpha,\beta)} M_\gamma.$$

For example, we have

$$M_1 \cdot M_1 = M_2 + 2M_{(1,1)},$$
$$M_1 \cdot M_2 = M_3 + M_{(1,2)} + M_{(2,1)},$$
$$M_{(1,2)} \cdot M_3 = M_{(1,2,3)} + M_{(1,3,2)} + M_{(3,1,2)} + M_{(1,5)} + M_{(4,2)}.$$

This product is commutative, though not in the graded sense. The coproduct is given by

$$\Delta\left(M_{(a_1,\dots,a_n)}\right) = \sum_{i=0}^{n} M_{(a_1,\dots,a_i)} \otimes M_{(a_{i+1},\dots,a_n)}.$$

For example,

$$\Delta\left(M_{(1,2)}\right) = 1 \otimes M_{(1,2)} + M_1 \otimes M_2 + M_{(1,2)} \otimes 1.$$

In particular, $\mathcal{Q}$Sym is not cocommutative.

There is a simple description of the antipode in this basis. To describe it, we first introduce a partial ordering on the set of compositions of n: say that

(2.31) $\quad \alpha < \beta \quad$ if α is obtained from the composition β by combining some adjacent parts.

A justification for this order relation is

Lemma 2.32. *Subsets of $\{1, \dots, n-1\}$ are in bijection with compositions of n.*

The idea is that if $\{x_1, \dots, x_k\}$ is a subset, written in increasing order, then $(x_1, x_2 - x_1, x_3 - x_2, \dots, n - x_k)$ is a composition of n. Now the partial order we introduced on the set of compositions corresponds to the partial order of $\mathcal{P}\{1, \dots, n-1\}$ given by subset inclusion.

Also, if $\beta = (b_1, \dots, b_k)$ is a composition of b, then the **reverse composition** is $\text{rev}\beta = (b_k, \dots, b_1)$.

Proposition 2.33 (Ehrenborg [E], Malvenuto [M]). *The antipode is given by*

$$\chi(M_\alpha) = (-1)^\alpha \sum_{\beta \le \alpha} M_{\text{rev}\beta}.$$

Example 2.34.

$$\chi(M_n) = -M_n, \qquad \chi(M_{(a,b)}) = M_{(a+b)} + M_{(b,a)}$$
$$\chi(M_{(a,b,c)} = -M_{(c,b,a)} - M_{(c+b,a)} - M_{(c,b+a)} - M_{(a+b+c)}$$

Another choice of basis for $\mathcal{Q}\mathrm{Sym}$ is the so-called **Fundamental basis**. This is defined as

$$F_\alpha := \sum_{\beta \geq \alpha} M_\beta.$$

For example,

$$F_1 = M_1, \qquad F_2 = M_2 + M_{(1,1)}, \qquad F_3 = M_3 + M_{(2,1)} + M_{(1,2)} + M_{(1,1,1)}$$
$$F_{(2,1)} = M_{(2,1)} + M_{(1,1,1)}$$

Remark 2.35. We can think of the formula defining the Fundamental basis in terms of the Monomial basis as saying that the change-of-basis matrix is upper-triangular, with 1's above the diagonal. This means that the reverse change-of-basis is pretty simple (see Möbius inversion on HW3). For example,

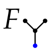
$$M_2 = F_2 - F_{(1,1)}, \qquad M_3 = F_3 - F_{(2,1)} - F_{(1,2)} + F_{(1,1,1)}.$$

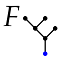
Wed, Feb. 15

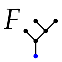
2.2. **Relating to the Hopf algebra of trees.** Given a tree $T \in \mathcal{T}_n$, label the leaves 0 through n, moving from left to right. Then let $L(T) \subseteq \{1, \ldots, n-1\}$ be the set of leaves (excluding the outer ones) which are **L**eft-pointing. We will abuse notation, using Lemma 2.32, and write $L(T)$ for the associated composition.

Proposition 2.36 ([AS, Thm 1.5]). *Define $\mathcal{T}\mathrm{Sym} \xrightarrow{\mathcal{L}} \mathcal{Q}\mathrm{Sym}$ by $\mathcal{L}(F_T) = F_{LT}$. Then $\mathcal{L}$ is a surjective map of Hopf algebras.*

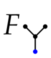
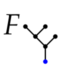
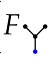
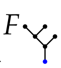
Example 2.37. We have

F  $\mapsto F_1 = M_1$

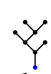
F  $\mapsto F_2 = M_2 + M_{(1,1)}$

F  $\mapsto F_{(1,1)} = M_{(1,1)}$

Exercise 2.38. Check that

$\mathcal{L}\Big(F$  $\cdot F$  $\Big) = \mathcal{L}\Big(F$  $\Big) \cdot \mathcal{L}\Big(F$  $\Big)$

Remark 2.39. We can use the change-of-basis relating the Monomial basis to the Fundamental basis in $\mathcal{Q}\mathrm{Sym}$ in order to define an analogue of the Monomial basis in $\mathcal{T}\mathrm{Sym}$. This requires a choice of partial order on the set of planar, binary trees (see the Tamari order in §1.1 of [AS]). Using this basis, it can be shown that the Hopf algebra surjection $\mathcal{L}$ is given by

$\mathcal{L}(M_T) = \begin{cases} M_{L(T)} & \text{(tree)} \not\subset T \\ 0 & \text{else} \end{cases}$ 

See [GR, Prop 5.23] for the Hopf algebra structure expressed in this basis.

Example 2.40. (The Leibniz-Hopf algebra) Just as we considered the dual pair $(\mathcal{T}, \mathcal{T}\mathrm{Sym})$ of Hopf algebras, we here consider the dual to $\mathcal{Q}\,\mathrm{Sym}$. This has a number of equivalent descriptions, and is variously called the **Leibniz-Hopf Algebra**, **Solomon's Descent algebra** Sol, and the algebra $\mathcal{N}\mathrm{Sym}$ of **noncommutative symmetric functions**.

Let $\{H_\alpha\}$ be the basis of $\mathcal{N}\mathrm{Sym}$ dual to the monomial basis of $\mathcal{Q}\,\mathrm{Sym}$.

Proposition 2.41. *The Hopf algebra $\mathcal{N}\mathrm{Sym}$ is isomorphic to the tensor algebra*

$$\mathcal{N}\mathrm{Sym} \cong T(H_1, H_2, H_3, \dots),$$

with coproduct on the algebra generators given by

$$\Delta(H_n) = \sum_{i+j=n} H_i \otimes H_j,$$

where $H_0 = 1$ is the identity element.

Proof. First recall the comultiplication in the monomial basis in $\mathcal{Q}\mathrm{Sym}$: writing $\beta \cdot \gamma$ for the **concatenation** of two compositions, we saw that

$$\Delta(M_\alpha) = \sum_{\alpha=\beta\cdot\gamma} M_\beta \otimes M_\gamma.$$

This implies that, in the dual, we have

$$H_\beta \cdot H_\gamma = H_{\beta\cdot\gamma}.$$

Since every composition is a (unique) concatenation of singletons, it follows that each basis element H_α is uniquely expressible as a product of H_n's, so that $\mathcal{N}\mathrm{Sym}$ is the tensor algebra as claimed.

To determine the coproduct on the H_n's, we must determine how M_n can appear as a term in a product of monomial basis elements in $\mathcal{Q}\mathrm{Sym}$. But the only way to quasi-shuffle M_α and M_β to yield M_n is if both α and β are singletons that sum to n. This gives the desired coproduct on H_n. ∎

Proposition 2.42. *The antipode in $\mathcal{N}\mathrm{Sym}$ is given by*

$$\chi(H_\alpha) = \sum_{\beta \geq \alpha} (-1)^\beta H_{\mathrm{rev}\beta}.$$

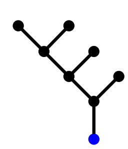
Proof. In general, if $\mathcal{H}$ is a Hopf algebra and $\mathcal{D}$ is the dual Hopf algebra, then for $d \in \mathcal{D}$, the antipode can be computed as $\chi_{\mathcal{D}}(d) = d \circ \chi_{\mathcal{H}}$. So given H_α, we wish to know the coefficient of $H_{\mathrm{rev}\beta}$ in $\chi_{\mathcal{N}}(H_\alpha)$. We determine this coefficient by evaluating on $M_{\mathrm{rev}\beta}$, and we find

$$\chi_{\mathcal{N}}(H_\alpha)(M_{\mathrm{rev}\beta}) = H_\alpha(\chi_{\mathcal{Q}} M_{\mathrm{rev}\beta}) = \sum_{\gamma \leq \mathrm{rev}\beta} (-1)^\beta \delta_{\alpha,\mathrm{rev}(\gamma)} = \begin{cases} (-1)^\beta & \mathrm{rev}\alpha \leq \mathrm{rev}\beta \\ 0 & \text{else} \end{cases}$$

∎

Fri, Feb. 17

By dualizing Proposition 2.36, we get a Hopf algebra morphism in the other direction. In order to describe it, let us denote by R_a the tree the a internal nodes, all on the left branch (this is called a right comb). For example, R_3 is

$R_3 =$ 

Proposition 2.43. *There is an injection of Hopf algebras* $\mathcal{N}\mathrm{Sym} \xrightarrow{\mathcal{R}} \mathcal{T}$ *given on the generators by*

$$\mathcal{R}(H_n) = R_n.$$

Proof. By dualizing the map of Proposition 2.36, we get an injection of Hopf algebras as claimed. It remains to explicitly determine the map $\mathcal{R}$.

Since $\mathcal{N}\mathrm{Sym}$ is a tensor algebra (free associative algebra), it suffices to specify it on the algebra generators, which are the H_n. Recall that H_n was defined to be the dual basis element to M_n. Unfortunately, the adjoint map $\mathcal{L}$ was defined using the Fundamental basis of $\mathcal{Q}\mathrm{Sym}$ rather than the monomial basis. In order to determine $\mathcal{R}(H_n)$, we must find all trees T such that M_n shows up as a term in $\mathcal{L}(F_T)$. However, since the singleton composition (n) was minimal in our partial order, this means that M_n is a term of F_n and of *no other* Fundamental basis element in degree n. Thus it suffices to find all trees T for which F_n shows up as a term in $\mathcal{L}(F_T)$. But $\mathcal{L}(F_T) = F_{L(T)}$, by definition, and recall that the singleton composition (n) corresponds to the subset $\emptyset \subseteq [n-1]$. Thus the required tree T has *none* of its leaves pointing to the left, which means it must be the right comb. ■

Exercise 2.44. Verify that the assignment $\mathcal{R}(H_n) = R_n$ preserves coproducts.

2.3. **Hopf ideals, quotient Hopf algebras.** The next example we will give will be a quotient of $\mathcal{N}\mathrm{Sym}$, but we first need to discuss how to make sense of quotient Hopf algebras.

Recall that given an algebra A, and an *ideal* $I \subseteq A$, we can make sense of a quotient algebra A/I, and we get a surjective ring map $A \twoheadrightarrow A/I$. On the other hand, we think of group algebras as typical examples of Hopf algebras, and there we know that we need a *normal* subgroup $H \leq G$ in order to make sense of a quotient group G/H.

Definition 2.45. Suppose that the ground ring R is a field. A Hopf algebra morphism $A \longrightarrow B$ is called **normal** if the two compositions

$$I_A \otimes B \longrightarrow B \otimes B \xrightarrow{\mu} B, \qquad B \otimes I_A \longrightarrow B \otimes B \xrightarrow{\mu} B$$

have the same image in B. Dually, we say that the map of Hopf algebras is **conormal** if the kernels of the two maps

$$A \xrightarrow{\Delta} A \otimes A \longrightarrow J_B \otimes A, \qquad A \xrightarrow{\Delta} A \otimes A \longrightarrow A \otimes J_B$$

agree, where $J_B := \mathrm{coker}(R \xrightarrow{\eta} B)$.

We have written things in this way for symmetry, but actually the composition $I_B \hookrightarrow B \twoheadrightarrow J_B$ is an isomorphism. Using the isomorphism $J_B \cong I_B$, the surjection $B \twoheadrightarrow I_B$ is $b \mapsto b - \varepsilon(b) \cdot 1$.

Mon, Feb. 20

Note that another way to describe the normality condition is to say that the left ideal in B generated by I_A agrees with the right ideal generated by I_A.

Example 2.46. Let $H \trianglelefteq G$ be a normal subgroup of a finite group. Then the restriction $R^G \longrightarrow R^H$ along the inclusion of H in G is a conormal Hopf algebra map. To see this, note first that $J_{R^H} = R^H/R$, the quotient of R^H by the subspace of constant functions. Given the identification $R^H \otimes R^G \cong R^{H\times G}$, it follows that we can identify $J_{R^H} \otimes R^G \cong R^{H\times G}/(\text{left } H\text{-invariant functions})$. Thus the kernel of $R^G \longrightarrow J_{R^H} \otimes R^G$ can be identified with the space of left H-invariant functions on G.

Similarly, the other kernel is the space of right-H-invariant functions on G. But if $f \in R^G$ is left-H-invariant, then

$$f(g \cdot h) = f(g \cdot h \cdot g^{-1} \cdot g) = f(h' \cdot g) = f(g),$$

so f is equivalently right-H-invariant. This shows that $R^G \longrightarrow R^H$ is conormal.

Remark 2.47. If R is not a field, then the definition of (co)normal needs an extra hypothesis that the quotient $B \longrightarrow B/(I_A \cdot B)$ is a split surjection. But a surjection of vector spaces always has a splitting, so this condition can be ommited when R is a field.

If $A \xrightarrow{f} B$ is normal, we denote by $B/\!/f$ the quotient

$$B/\!/f := B/(I_A \cdot B) = B/(B \cdot I_A).$$

This can also be described as $B/\!/f \cong R \otimes_A B \cong B \otimes_A R$. When f is an inclusion, we typically write $B/\!/A$ for the quotient.

Proposition 2.48. *Suppose that $A \xrightarrow{f} B$ is a normal map of Hopf algebras. Then $B/\!/f$ inherits a Hopf algebra structure, and the quotient $B \longrightarrow B/\!/f$ is a Hopf algebra map.*

Proof. We must define η, μ, ε, Δ, and χ. The unit is easy, as we can just use the unit of B, together with the quotient $B \longrightarrow B/\!/f$. In order to see that the multiplication in B descends to a multiplication in $B/\!/f$, we must check that if $\{b_i\}$ are elements of B and $\{a_j\}$ are elements of I_A, then the combinations $\sum_{i,j} b_i a_j$ and $\sum_{i,j} a_j b_i$ lie in $B \cdot I_A = I_A \cdot B$. But this holds, by the normality condition.

The counit is induced by ε_B. More precisely, since f is a Hopf algebra map, we know that $f(I_A) \subseteq I_B$, so that we can take the counit to be $B/(I_A \cdot B) \longrightarrow B/(I_B \cdot B) \cong R$. For the comultiplication, it suffices to see that

$$B \xrightarrow{\Delta} B \otimes B \longrightarrow B/\!/f \otimes B/\!/f$$

sends $I_A \cdot B$ to zero. But Lemma 2.49 implies that $\Delta(I_A \cdot B) \subseteq I_A \cdot B \otimes B + B \otimes I_A \cdot B$, which gets collapsed in $B/\!/f \otimes B/\!/f$. ■

Lemma 2.49. *Let A be a bialgebra. Then $\Delta(I_A) \subseteq I_A \otimes A + A \otimes I_A$.*

Proof. The main idea is to use the splitting $A \cong I_A \oplus R$ given by $a \mapsto (a - \varepsilon(a) \cdot 1, \varepsilon(a))$. This gives an isomorphism

$$A \otimes A \cong (R \otimes R) \oplus (I_A \otimes A + A \otimes I_A),$$

so that $I_A \otimes A + A \otimes I_A = \ker(A \otimes A \xrightarrow{\varepsilon\otimes\varepsilon} R \otimes R)$. Now the square

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & A \otimes A \arrow[d, "\varepsilon\otimes\varepsilon"] \\ R \arrow[r, "\cong"'] & R \otimes R \end{tikzcd}$$

commutes, which implies the statement of the lemma ■

Wed, Feb. 22

Of course, we have a dual result in the conormal context. If $A \xrightarrow{g} B$ is conormal, we write $A\backslash\!\!\backslash g$ for the kernel

$$A\backslash\!\!\backslash g := \ker(A \xrightarrow{\Delta} A \otimes A \longrightarrow J_B \otimes A) \cong \ker(A \xrightarrow{\Delta} A \otimes A \longrightarrow A \otimes J_B).$$

Proposition 2.50. *Suppose that $A \longrightarrow B$ is a conormal map of Hopf algebras. Then $A\backslash\!\!\backslash g$ inherits a Hopf algebra structure, and the inclusion $A\backslash\!\!\backslash g \longrightarrow A$ is a Hopf algebra map.*

Proposition 2.48 gives a way of producing new Hopf algebras in analogy with what we do in the world of groups. On the other hand, this is not quite what we do when forming quotient algebras: there, we start just with an *ideal* $I \subseteq A$ rather than a subalgebra. There is a similar notion for Hopf algebras, generalizing the property we saw in Lemma 2.49.

Definition 2.51. A two-sided ideal $L \subseteq I_A$ inside a Hopf algebra is called a **Hopf ideal** if

$$\Delta(L) \subseteq L \otimes A + A \otimes L.$$

We then get

Proposition 2.52. *If $L \subseteq A$ is a Hopf ideal, then A/L inherits a Hopf algebra structure, and the quotient $A \twoheadrightarrow A/L$ is a surjection of Hopf algebras.*

In the course of Proposition 2.48, we showed that if $A \xrightarrow{f} B$ is a normal map of Hopf algebras, then $I_A \cdot B \subseteq B$ is a Hopf ideal.

Example 2.53. Let B be a commutative Hopf algebra, and $y \in B$ a primitive element. Then $L = (y)$ is a Hopf ideal, and we have a surjection of Hopf algebras $B \longrightarrow B/y$.

Example 2.54. For instance, take $B = \mathbb{F}_2[x_1]$ and $y = x_1^{2^n}$. We then have the Hopf quotient

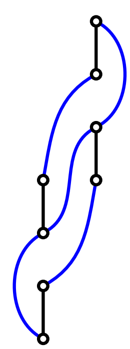
$$\mathbb{F}_2[x_1] \twoheadrightarrow \mathbb{F}_2[x_1]/(x_1^{2^n}).$$

On the other hand, x_1^3 does not generate a Hopf ideal, and correspondingly $\mathbb{F}_2[x_1]/x_1^3$ is not a Hopf algebra.

Example 2.55. In $B = T_{\mathbb{F}_2}(x_1, x_2)$ with $\overline{\Delta}(x_2) = x_1 \otimes x_1$, consider the 2-sided ideal L generated by $(x_1^2, x_2^2 + x_1x_2x_1)$. We claim this is a Hopf ideal. x_1^2 is primitive, but showing that $\Delta(x_2^2 + x_1x_2x_1) \in L \otimes B + B \otimes L$ is not immediate.

Assuming the claim, it follows that the quotient $T_{\mathbb{F}_2}(x_1, x_2)/(x_1^2, x_2^2 + x_1x_2x_1)$ is a Hopf algebra. We claim it is finite-dimensional.

basis	degree
1	0
x_1	1
x_2	2
$x_1x_2,\ x_2x_1$	3
x_2^2	4
$x_2x_1x_2$	5
x_2^3	6



In the figure, a black segment (going up) denotes left multiplication by x_1, and the blue curves denote left multiplication by x_2.

Example 3.1. (The mod 2 Steenrod algebra) The Steenrod algebra will be a quotient of the Leibniz-Hopf algebra, working over $\mathbb{F}_2$. We define $L_{\text{Adem}} \subseteq \mathcal{N}\text{Sym}_{\mathbb{F}_2}$ to be the two-sided ideal generated by the elements

$$r_{i,j} := H_i H_j + \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} H_{i+j-k} H_k, \qquad \text{for} \quad 0 < i < 2j.$$

The relations generating L are known as the **Adem relations**. We define the **mod 2 Steenrod algebra** to be the quotient

$$\mathcal{A} := \mathcal{N}\text{Sym}_{\mathbb{F}_2} / L_{\text{Adem}}.$$

We will denote by $\boxed{i}$ the image of H_i in $\mathcal{A}$. This is pronounced "square i" and usually written as Sq^i. These elements are known as the "Steenrod squares".

Example 3.2. Taking $i = j = 1$, we have the Adem relation

$$\boxed{1}\boxed{1} = 0 \cdot \boxed{2} = 0,$$

so that $\boxed{1}$ becomes an exterior element in $\mathcal{A}$. Taking $i = 1$ and $j = 2$, we have

$$\boxed{1}\boxed{2} = 1 \cdot \boxed{3} = \boxed{3},$$

so that the generator H_3 becomes decomposable in $\mathcal{A}$.

Fri, Feb. 24

Exercise 3.3. Show that $\boxed{1}\boxed{2n} = \boxed{2n+1}$ for any n.

Exercise 3.4. Show that $\boxed{2n-1}\boxed{n} = 0$.

If $I = (i_1, i_2, \ldots, i_k)$ is a sequence of nonnegative integers, we will write $\boxed{I}$ for the product $\boxed{i_1}\boxed{i_2} \ldots \boxed{i_k}$. Say that a monomial $\boxed{I}$ is **admissible** if $i_r \geq 2i_{r+1}$ for all $r < k$ and $i_k \geq 1$. We also include $\boxed{0} = 1$ as an admissible monomial. Then one interpretation of the Adem relations is that they express an inadmissible monomial $\boxed{i}\boxed{j}$ as a sum of admissible monomials. For example, $\boxed{2}\boxed{2}$ can be written as the admissible monomial $\boxed{3}\boxed{1}$. We will show below that the admissible monomials give a basis for $\mathcal{A}$. We will largely follow [SE] in this discussion.

Proposition 3.5. *The admissible monomials span $\mathcal{A}$.*

Proof. For monomials of length 2, this is exactly given by the Adem relations. What is not immediate is that repeatedly applying Adem relations to a longer monomial will eventually produce a sum of admissible monomials. The key is to introduce the **moment** of $\boxed{I}$, which is defined as

$$m(\boxed{I}) = i_1 + 2i_2 + \cdots + ki_k.$$

Then each application of an Adem relation *strictly* reduces the moment. It follows that the Adem relations can only be applied to $\boxed{I}$ finitely many times, meaning that eventually all terms will be admissible. ■

For example, applying Adem relations first on the left and then on the right gives

$$\boxed{2}\boxed{4}\boxed{5} = \boxed{6}\boxed{5} + \boxed{5}\boxed{1}\boxed{1} = \boxed{6}\boxed{5} = \binom{4}{6}\boxed{11} + \binom{3}{4}\boxed{10}\boxed{1} + \binom{2}{2}\boxed{9}\boxed{2} + \binom{1}{0}\boxed{8}\boxed{3}$$
$$= \boxed{9}\boxed{2} + \boxed{8}\boxed{3}$$

Mon, Feb. 27

In order to show linear independence, we will apply a linear map to a polynomial ring and show that the results are independent.

One tool that we will employ is the action of $\mathcal{A}$ on polynomial algebras.

Proposition 3.6. *Let $P_n = \mathbb{F}[x]^{\otimes n}$, where $\deg(x) = 1$. Then there is a (left) action of $\mathcal{A}$ on P_n, determined by the following:*

(1) $\boxed{k}(x^n) = \binom{n}{k}x^{n+k}$
(2) *(Cartan formula) For polynomials $p(\mathbf{x}), q(\mathbf{x})$, we have* $\boxed{k}(p(\mathbf{x})q(\mathbf{x})) = \sum_{i+j=k}\boxed{i}(p(\mathbf{x}))\boxed{j}(q(\mathbf{x}))$.

We will not prove Proposition 3.6, though it is a special case of Proposition 3.16 (which we will also not prove).

An important reinterpretation of the Cartan formula is that the diagram

$$\begin{tikzcd}
\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes P_n \otimes P_n \arrow[r, "\Delta\otimes\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes\mu"'] & \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes P_n \otimes P_n \arrow[r, "\cong"] & \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes P_n \otimes \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes P_n \arrow[d] \\
 & & P_n \otimes P_n \arrow[d, "\mu"] \\
\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes P_n \arrow[rr] & & P_n
\end{tikzcd}$$

commutes. The same diagram also commutes when replacing $\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2}$ by $\mathcal{A}$, though we have not yet shown that Δ descends to $\mathcal{A}$.

Let us denote by $\sigma_n = x_1x_2\dots x_n \in \mathbb{F}[x_1, \dots, x_n]$, the nth elementary symmetric function.

Proposition 3.7. *The function $\mathcal{A} \xrightarrow{\omega_n} P_n$ defined by $\omega_n(\theta) = \theta(\sigma_n)$ sends admissible sends the admissible monomials in degree $\leq n$ to linearly independent polynomials.*

Corollary 3.8. *The function $\mathcal{A} \xrightarrow{\omega_n} P_n$ defined by $\omega_n(\theta) = \theta(\sigma_n)$ is injective in degrees $\leq n$.*

We will see later that ω_n vanishes in degree larger than n, so this result is optimal.

We will prove Proposition 3.7 now, though we handle one preliminary result first. Since we will often need to compute binomial coefficients mod 2, recall that these can be easily computed: Lucas's theorem states that is $a = a_n2^n + a_{n-1}2^{n-1} + \dots a_0$ and $b = b_k2^k + \dots + b_0$, with $a_i, b_i \in \{0,1\}$, then

$$\binom{a}{b} \equiv \prod_i \binom{a_i}{b_i} \pmod 2.$$

In particular, since each $\binom{a_i}{b_i}$ is zero only if $a_i = 0$ and $b_i = 1$, it follows that $\binom{a}{b}$ is zero exactly when b contains some 2-adic digit that is not in a. In particular, the formula $\boxed{k}(x^n) = \binom{n}{k}x^{n+k}$ yields

Lemma 3.9. $\boxed{i}(x^{2^n})$ *is nonzero only for $i = 0, 2^n$.*

Proof of Proposition 3.7. The proof is by induction on n. The base case $n = 1$ is verified, since $\boxed{0}(x) = x$ and $\boxed{1}(x) = x^2$.

So now suppose the statement holds in the case of $n-1$. Now consider some linear combination

$$\sum_{I \text{ adm}} a_I \boxed{I}(\sigma_n) = 0$$

where the sum is over admissible monomials of some fixed degree $q \leq n$. We wish to show that $a_I = 0$ for all such I. By (downward) induction on the length of I, it suffices to show that $a_I = 0$ for all I of largest length $\ell(I) = m$.

Suppose, for instance, that we know that a_I vanishes for monomials of length 4 or higher. Then we can write our relation as

$$\sum_{\substack{I \text{ adm} \\ \ell(I)=3}} a_I \boxed{I}(\sigma_n) + \sum_{\substack{I \text{ adm} \\ \ell(I)\leq 2}} a_I \boxed{I}(\sigma_n) = 0. \tag{3.10}$$

Wed, Mar. 1

For the induction step, we wish to show that all a_I vanish if $\ell(I) = 3$. By the Cartan formula, we have

$$\boxed{I}(x_1 \cdots x_n) = \sum_{J \leq I} \boxed{J}(x_1)\boxed{I-J}(x_2 \cdots x_n),$$

where the comparison $J \leq I$ means pointwise (i.e. $j_r \leq i_r$ for all r). Let $\varphi : \mathbb{F}_2[x_1, \ldots, x_n] \longrightarrow \{x_1^8\} \cdot \mathbb{F}_2[x_2, \ldots, x_n]$ be the projection. Since $\boxed{J}(x_1) = x_1^8$ only if $J = (4,2,1)$, by Lemma 3.9, then applying φ to (3.10) gives

$$\sum_{\substack{I \text{ adm} \\ \ell(I)=3}} a_I x_1^8 \boxed{I - (4,2,1)}(\sigma_{n-1}) = 0$$

Each $I - (4,2,1)$ is admissible. In fact, subtracting $(4,2,1)$ gives a bijection between admissible monomials of length 3 and degree q to admissible monomials of length at most 3 and degree $q-7$. But now by the inductive hypothesis on n, we conclude that each a_I must be zero. ∎

Proposition 3.7 implies that the admissible monomials are linearly independent. Thus we have shown

Theorem 3.11 (Serre)**.** *The admissible monomials give a basis of $\mathcal{A}$.*

We now show that $\mathcal{A}$ is a Hopf algebra.

Theorem 3.12 (Milnor, 1958)**.** *The ideal L is a Hopf ideal in $\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2}$, so that $\mathcal{A}$ becomes a Hopf algebra.*

Proof of Theorem 3.12. We wish to show that the composition

$$L_{\text{Adem}} \longrightarrow \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \xrightarrow{\Delta} \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \xrightarrow{q\otimes q} \mathcal{A} \otimes \mathcal{A}$$

vanishes. Fix an (arbitrary) n. We will show that the composition vanishes in degrees $\leq n$. Consider the diagram

$$\begin{tikzcd}
\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \arrow[r, "\Delta"] \arrow[d, "q"'] & \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \otimes \mathcal{N}\mathrm{Sym}_{\mathbb{F}_2} \arrow[r, "q\otimes q"] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\omega_n \otimes \omega_n"] \\
\mathcal{A} \arrow[rr, "\omega_n \times \omega_n"] & & \mathbb{F}[x_1, \ldots, x_n, y_1, \otimes, y_n],
\end{tikzcd}$$

where the map labeled $\omega_n \times \omega_n$ is $\theta \mapsto \theta(\sigma_n(\mathbf{x})\sigma_n(\mathbf{y}))$. Again, the Cartan formula says that this diagram commutes. If we restrict the (down, right) composition to L_{Adem}, it vanishes by the definition of $\mathcal{A}$. It follows that the clockwise composition also vanishes on L_{Adem}. Since $\omega_n \otimes \omega_n$ is injective by Proposition 3.8, we get the desired conclusion. ∎

Fri, Mar. 3

As $\mathcal{A}$ is a quotient of $\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2}$, the antipode formula (Prop 2.42) carries over to $\mathcal{A}$:

Proposition 3.13. *The antipode in $\mathcal{A}$ is given on monomials by*

$$\chi(\boxed{I}) = \sum_{\substack{J \in \mathrm{Comp}(d(I)) \\ J \geq_c I}} \boxed{\mathrm{rev} J},$$

where the ordering is the ordering on compositions given in (2.31).

For example, since the singleton composition is minimal in the partial order on compositions, this implies that the antipode on $\boxed{n}$ is the sum of the square's on all compositions of n. But note that many of these squares will be inadmissible, so this result is not expressed in the basis.

Example 3.14.

$$\chi(\boxed{1}) = \boxed{1}.$$
$$\chi(\boxed{2}) = \boxed{2} + \boxed{1}^2 = \boxed{2}.$$
$$\chi(\boxed{3}) = \boxed{3} + \boxed{2}\boxed{1} + \boxed{1}\boxed{2} + \boxed{1}^3 = \boxed{2}\boxed{1}.$$

Note that the last computation agrees with the identity $\boxed{3} = \boxed{1}\boxed{2}$, so that $\chi(\boxed{3}) = \chi(\boxed{2})\chi(\boxed{1})$.

Proposition 3.15. *The element $\boxed{i}$ is decomposable if and only if i is not a power of 2. In other words, $\mathcal{A}$ is generated by the indecomposable elements $\{\boxed{1}, \boxed{2}, \boxed{4}, \boxed{8}, \dots\}$.*

Proof. We saw already that, in the polynomial ring $\mathbb{F}_2[x]$, the only nontrivial operations on x^{2^n} are $\boxed{0}$ and $\boxed{2^n}$. It follows that $\boxed{2^n}$ is indecomposable.

To show that the other squares are decomposable, consider some $0 < i < 2^n$. Then the Adem relation gives

$$\boxed{i}\boxed{2^n} = \binom{2^n - 1}{i}\boxed{2^n + i} + \text{other (decomposable) terms},$$

which shows that $\boxed{2^n + i}$ is decomposable, as the binomial coefficient is nonzero. ∎

We saw before that $\mathcal{A}$ acts on a polynomial algebra $\mathbb{F}_2[x_1, \dots, x_n]$. Recall that this polynomial algebra can be identified with the cohomology ring $\mathrm{H}^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty; \mathbb{F}_2)$. More generally, we have

Proposition 3.16. *The Steenrod algebra acts naturally on $\mathrm{H}^*(X; \mathbb{F}_2)$. Moreover, this action satisfies*

(1) *It is stable under suspension: for every i and n, the diagram*

$$\begin{tikzcd} \tilde{\mathrm{H}}^n(X;\mathbb{F}_2) \arrow[r, "\boxed{i}"] \arrow[d, "\sigma"', "\cong"] & \tilde{\mathrm{H}}^{n+i}(X;\mathbb{F}_2) \arrow[d, "\cong"', "\sigma"] \\ \tilde{\mathrm{H}}^{n+1}(\Sigma X;\mathbb{F}_2) \arrow[r, "\boxed{i}"] & \tilde{\mathrm{H}}^{n+i+1}(X;\mathbb{F}_2) \end{tikzcd}$$

commutes.

(2) *If $x \in \mathrm{H}^n(X; \mathbb{F}_2)$, then $\boxed{n}(x) = x^2$.*

(3) *If $x \in \mathrm{H}^n(X; \mathbb{F}_2)$ and $i > n$, then $\boxed{i}(x) = 0$.*

This action makes $\mathrm{H}^*(X;\mathbb{F}_2)$ *into an* $\mathcal{A}$*-algebra, meaning that the unit and multiplication of* $\mathrm{H}^*(X;\mathbb{F}_2)$ *are both* $\mathcal{A}$*-module maps.*

We unravel the last statement. In order to state that the cup product is an $\mathcal{A}$-module map, we need to describe $\mathrm{H}*(X)\otimes \mathrm{H}^*(X)$ as an $\mathcal{A}$-module. The module structure is given by

$$\mathcal{A}\otimes \mathrm{H}^*(X)\otimes \mathrm{H}^*(X) \xrightarrow{\Delta} \mathcal{A}\otimes\mathcal{A}\otimes \mathrm{H}^*(X)\otimes \mathrm{H}^*(X) \cong \mathcal{A}\otimes \mathrm{H}^*(X)\otimes\mathcal{A}\otimes \mathrm{H}^*(X) \xrightarrow{a\otimes a} \mathrm{H}^*(X)\otimes \mathrm{H}^*(X).$$

Given the description of the coproduct on $\mathcal{A}$, this means that

$$\boxed{n}\cdot(x\otimes y) = \sum_{i+j=n} \boxed{i}(x)\otimes\boxed{j}(y).$$

Thus to say that the multiplication is an $\mathcal{A}$-module map means that we have the formula

$$\text{(Cartan)} \qquad \boxed{n}(x\cdot y) = \sum_{i+j=n} \boxed{i}(x)\cdot\boxed{j}(y).$$

This is also represented as in the diagram after Proposition 3.6.

Corollary 3.17. *Suppose that* X *is a space such that* $\mathrm{H}^*(X;\mathbb{F}_2)\cong\mathbb{F}_2[x_n]/x_n^r$*, where* $r\in\{3,\dots,\infty\}$*. Then* $n=2^k$ *for some* k*.*

Proof. We know that $\boxed{n}x_n = x_n^2$. On the other hand, if n is not a power of 2, then $\boxed{n}$ is decomposable. Since there are no classes in the intermediate degrees between x_n and x_n^2, this implies that $\boxed{n}x_n = 0$. ■

Wed, Mar. 8

3.1. The Dual Steenrod algebra.

Definition 3.18. We define $\mathcal{A}_* := \mathcal{A}^\vee$ to be the dual Hopf algebra to $\mathcal{A}$.

As $\mathcal{A}$ was a cocommutative Hopf algebra, it follows that $\mathcal{A}_*$ is a *commutative* Hopf algebra. Since $\mathcal{A}$ was a quotient of $\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2}$, it follows that $\mathcal{A}_*$ is a sub-Hopf-algebra of $\mathcal{Q}\mathrm{Sym}_{\mathbb{F}_2}$.

Define $z_i\in\mathcal{A}_*$ to be dual to $\boxed{2^{i-1},\dots,2,1}$. Thus $\deg(z_i)=2^i-1$. The elements z_i are typically written ζ_i or ξ_i.

Proposition 3.19. *Under the inclusion* $\mathcal{A}_*\hookrightarrow\mathcal{Q}\mathrm{Sym}_{\mathbb{F}_2}$*, the element* z_n *maps to* $M_{(2^{n-1},2^{n-2},\dots,2,1)}$*.*

Proof. We must check that, in the dual quotient map $\mathcal{N}\mathrm{Sym}_{\mathbb{F}_2}\longrightarrow\mathcal{A}$, the admissible monomial $\boxed{2^{i-1},\dots,2,1}$ does not show up as a term in the image of any other basis element H_I. If I is admissible, this follows from the fact that the $\boxed{I}$, with I admissible, form a basis. Thus suppose that I is inadmissible. In particular, suppose that I contains the substring (i,j), where $i<2j$. It suffices to show that applying the Adem relations to $\boxed{i,j}$ does not produce a term of the form $\boxed{2^k}$ or $\boxed{2^k,2^{k-1}}$.

We know that $\boxed{2^k}$ cannot appear as a term in $\boxed{i,j}$, since $\boxed{2^k}$ is indecomposable. So let's rule out terms of the form $\boxed{2^k,2^{k-1}}$. Consider the (potential) Adem relation

$$\boxed{i,j} = \boxed{2^k,2^{k-1}} + \text{other terms}.$$

In order for this to occur, we must have $2^{k-1}\le\lfloor\frac{i}{2}\rfloor$, so that $2^k\le i$. But if $i+j=2^k+2^{k-1}$, this forces $j\le 2^{k-1}$, so that $2j\le 2^k\le i$, meaning that (i,j) is admissible. ■

Recall (Example 2.30) that in $\mathcal{Q}$Sym, we have $M_1^2 = M_2 + 2M_{1,1} \equiv M_2$. In particular, $z_1^2 \neq 0$. Similarly, $M_1 \cdot M_2 = M_3 + M_{(1,2)} + M_{(2,1)}$, so $z_1^3 \neq 0$. By induction, you can show that $z_1^k \neq 0$ for all k. This is a special case of the following remarkable result.

Theorem 3.20 (Milnor, [M]). *The dual Steenrod algebra $\mathcal{A}_*$ is a polynomial algebra,*

$$\mathcal{A}_* \cong \mathbb{F}_2[z_1, z_2, \ldots],$$

where z_i is dual to $\boxed{2^{i-1}, \ldots, 2, 1}$ and has degree $2^i - 1$. The coproduct is given by

$$\Delta(z_n) = \sum_{0 \le k \le n} z_{n-k}^{2^k} \otimes z_k,$$

and

$$\chi(z_n) = \sum_{\alpha \in \mathrm{Comp}(n)} z_{a_1}^{2^{\sigma(1)}} \cdots z_{a_\ell(\alpha)}^{2^{\sigma(\ell(\alpha))}},$$

where $\sigma(k) = \sum_{j<k} a_j$

Proof sketch. For an admissible sequence $I = (i_1, \ldots, i_k)$, denote by $\gamma(I) = (i_1 - 2i_2, i_2 - 2i_3, \ldots, i_k)$ the sequence of excesses. Note that as I runs over the admissible sequences, $\gamma(I)$ runs over the (finite) sequences of nonnegative integers. Ordering the admissible sequences by *right* lexicographic ordering, the main step is to show that, for admissible sequences I and J,

$$\langle z^{\gamma(J)}, \boxed{I} \rangle = \begin{cases} 1 & I = J \\ 0 & I < J. \end{cases}$$

Since the $\boxed{I}$ give a basis for $\mathcal{A}$, it follows that the monomials $z^{\gamma(J)}$ give a basis for $\mathcal{A}_*$, which shows that $\mathcal{A}_*$ is a polynomial algebra.

We do not show this main step in full, but we carry out the argument in degree 4 as a demonstration. The two admissible sequences in degree 4 are $(3,1)$ and (4). We want to show that $z^{\gamma(3,1)} = z_1 z_2$ and $z^{\gamma(4)} = z_1^4$ are linearly independent. This will show they give a basis in degree 4, since we know (from $\mathcal{A}$) that this is a two-dimensional space. We evaluate these two monomials on the admissible squares in degree 4. The key for doing so is the following formula:

$$\langle x \cdot y, z \rangle = \langle \mu(x \otimes y), z \rangle = \langle x \otimes y, \Delta(z) \rangle = \sum_{(z)} \langle x, z_{(1)} \rangle \cdot \langle y, z_{(2)} \rangle.$$

For example,

$$\begin{aligned}
\langle z_1 z_2, \boxed{3,1} \rangle &= \langle z_1 \otimes z_2, \Delta(\boxed{3,1}) \rangle = \langle z_1 \otimes z_2, \sum_{J \le_p (3,1)} \boxed{J} \otimes \boxed{(3,1) - J} \rangle \\
&= \langle z_1, \boxed{1} \rangle \cdot \langle z_2, \boxed{2,1} \rangle = 1.
\end{aligned}$$

Similarly,

$$\langle z_1 z_2, \boxed{4} \rangle = \langle z_1, \boxed{1} \rangle \cdot \langle z_2, \boxed{3} \rangle = 1 \cdot 0 = 0.$$

On the other hand,

$$\langle z_1^4, \boxed{4} \rangle = \langle z_1, \boxed{1} \rangle \cdot \langle z_1^3, \boxed{3} \rangle = 1$$

by induction. It follows that $z_1 z_2$ and z_1^4 are independent. ∎

Mon, Mar 20

We next write down some quotient Hopf algebras of $\mathcal{A}_*$.

Example 3.21. Consider the exterior algebra $\mathcal{A}_*/(z_1^2, z_2, z_3, \dots) \cong \mathbb{F}_2[z_1]/(z_1^2)$. Note that once we set $z_2 = 0$, the coproduct formula $\overline{\Delta}(z_2) = z_1^2 \otimes z_1$ forces $z_1^2 = 0$. This quotient is often written either $\mathcal{A}(0)_*$ or $\mathcal{E}(0)_*$. Letting $\mathcal{A}(0)$ or $\mathcal{E}(0)$ denote the dual of $\mathcal{A}(0)_*$ (or $\mathcal{E}(0)_*$), we have an inclusion $\mathcal{A}(0) \hookrightarrow \mathcal{A}$, which realizes $\mathcal{A}(0)$ as the (exterior) subalgebra generated by $\boxed{1}$.

Example 3.22. On the recent worksheet, you were asked to consider $\mathcal{A}_*/(z_1^2, z_2^2, z_3, \dots) \cong E_{\mathbb{F}_2}(z_1, z_2)$. This quotient is usually written $\mathcal{E}(1)_*$. Again, we get an inclusion $\mathcal{E}(1) \hookrightarrow \mathcal{A}$ of the dual Hopf algebra. The class z_1, z_2, and z_1z_2 in $\mathcal{E}(1)_*$ dualize to give classes y_1, y_3, and y_4 in $\mathcal{E}(1)$. Since $y_1 \neq 0$ is in degree 1, we know it gets mapped to $\boxed{1} \in \mathcal{A}$. To understand the image of y_3, recall that y_3 was dual to z_2, and that the latter was dual to $\boxed{2,1}$. This means that

$$\langle z_2, \boxed{2,1} \rangle = 1, \qquad \langle z_2, \boxed{3} \rangle = 0.$$

Now the dual of z_2, which we will temporarily call w_3, should be an element of $\mathcal{A}^3$ characterized by

$$\langle z_2, w_3 \rangle = 1, \qquad \langle z_1^3, w_3 \rangle = 0.$$

However, as we saw last time, $\langle z_1^3, \boxed{2,1} \rangle = \langle z_1^2, \boxed{2} + \boxed{1,1} \rangle = 1$, which shows that $w_3 \neq \boxed{2,1}$. On the other hand, $\langle z_1^3, \boxed{3} \rangle = 1$, so that it follows that $w_3 = \boxed{2,1} + \boxed{3}$.

We could perform a similar analysis to determine the image of y_4 in $\mathcal{A}^4$. But in $\mathcal{E}(1)_*$, we have $\overline{\Delta}(z_1z_2) = z_1 \otimes z_2 + z_2 \otimes z_1$. This tells us that, in the dual, we must have $y_1 \cdot y_3 = y_4 = y_3 \cdot y_4$. It follows that

$$y_4 \mapsto \boxed{1}(\boxed{2,1} + \boxed{3}) = \boxed{3,1}.$$

The element $\boxed{2,1} + \boxed{3}$ is typically denoted Q_1, and we have just shown that $E(1) \cong E(Q_0, Q_1)$, where $Q_0 := \boxed{1}$. Furthermore, applying the coproduct formula shows that Q_1 is primitive.

Example 3.23. More generally, we have

$$\mathcal{E}(n)_* = \mathcal{A}_*/(z_1^2, \dots, z_{n+1}^2, z_{n+2}, \dots) \cong \bigotimes_{j=1}^{n+1} \mathbb{F}_2[z_j]/z_j^2.$$

Note that setting $z_2^2 = 0$ does not force $z_1^2 = 0$. The formula $\overline{\Delta}(z_2^2) = z_1^4 \otimes z_1^2$ forces $z_1^4 = 0$.

Example 3.24. Define

$$\mathcal{A}(1)_* = \mathcal{A}_*/(z_1^4, z_2^2, z_3, z_4, \dots) \cong \mathbb{F}_2[z_1, z_2]/(z_1^4, z_2^2).$$

This is an 8-dimensional algebra, with top degree element $z_1^3 z_2$, living in degree 6. The dual $\mathcal{A}(1) \subseteq \mathcal{A}$ turns out to be the subalgebra of $\mathcal{A}$ generated by $\boxed{1}$ and $\boxed{2}$ that we studied in example 2.55.

More generally, we define

$$\mathcal{A}(n)_* = \mathcal{A}_*/(z_1^{2^{n+1}}, z_2^{2^n}, \dots, z_{n+1}^2, z_{n+2}, \dots) \cong \mathbb{F}_2[z_1, \dots, z_{n+1}]/(z_1^{2^{n+1}}, z_2^{2^n}, \dots, z_{n+1}^2).$$

This is an algebra of dimension $2^{\frac{(n+1)(n+2)}{2}}$, with top degree element in degree $2^{n+2}(n-1) + n + 5$.

Proposition 3.25. *The dual $\mathcal{A}(n)$ of $\mathcal{A}(n)_*$ is the subalgebra of $\mathcal{A}$ generated by $\boxed{1}, \boxed{2}, \boxed{4}, \dots, \boxed{2^n}$.*

Corollary 3.26. *Each positive-degree element of $\mathcal{A}$ is nilpotent.*

$\boxed{\textbf{Wed, Mar. 22}}$

Proposition 3.27. *For a (finite type) Hopf algebra $\mathcal{H}$ over a field, the quotient $I_{\mathcal{H}} \xrightarrow{q} Q_{\mathcal{H}} = I_{\mathcal{H}}/I_{\mathcal{H}}^2$ induces an injection $(Q_{\mathcal{H}})^{\vee} \xrightarrow{q^{\vee}} \mathcal{H}^{\vee}$ whose image is $P(\mathcal{H}^{\vee})$.*

Proof. We have an exact sequence of vector spaces

$$I_{\mathcal{H}} \otimes I_{\mathcal{H}} \xrightarrow{\mu} I_{\mathcal{H}} \xrightarrow{q} Q_{\mathcal{H}} \longrightarrow 0.$$

As an exact sequence of vector spaces, it splits, so it follows that on passage to duals, the sequence

$$0 \longrightarrow (Q_{\mathcal{H}})^{\vee} \xrightarrow{q^{\vee}} I_{\mathcal{H}}^{\vee} \xrightarrow{\Delta} (I_{\mathcal{H}})^{\vee} \otimes (I_{\mathcal{H}})^{\vee}$$

is also (split) exact. ■

Example 3.28. The algebra generators $\boxed{2^n}$ in $\mathcal{A}$ correspond to the primitive elements $z_1^{2^n}$ of $\mathcal{A}_*$. Beware, however, that while it is true that $z_1^{2^n} = \boxed{2^n}^{\vee}$, it is **not** true generally that z_1^k is dual to $\boxed{k}$. For example,

$$z_1^3 = \boxed{2,1}^{\vee} + \boxed{3}^{\vee}, \qquad z_1^6 = \boxed{6}^{\vee} + \boxed{4,2}^{\vee}, \qquad z_1^7 = \boxed{7}^{\vee} + \boxed{6,1}^{\vee} + \boxed{5,2}^{\vee} + \boxed{4,2,1}^{\vee}.$$

Example 3.29. The algebra generators z_n in $\mathcal{A}_*$ give rise to primitive elements $Q_{n-1} \in \mathcal{A}^{2^n-1}$. The first few examples are

$$Q_0 = \boxed{1}, \qquad Q_1 = \boxed{2,1} + \boxed{3}, \qquad Q_2 = \boxed{4,2,1} + \boxed{5,2} + \boxed{6,1} + \boxed{7}$$

4. Cohomology of Hopf algebras

Let $\mathcal{H}$ be a Hopf algebra over a field k. Note that the counit $\mathcal{H} \xrightarrow{\varepsilon} k$ is an algebra map and thus makes k into an $\mathcal{H}$-module (in a rather trivial way). Note that if $\mathcal{H}$ is not commutative, we need to take care to distinguish between left and right modules. Let us consider k as a left $\mathcal{H}$-module.

Definition 4.1. Let $\mathcal{H}$ be a Hopf algebra over a field k. We then define $\mathrm{H}^n(\mathcal{H}) := \mathrm{Ext}^n_{\mathcal{H}}(k,k)$.

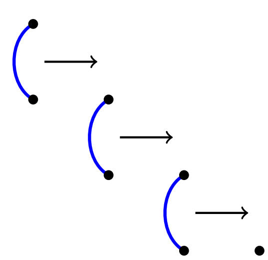
Recall that one way to compute Ext is as follows: let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k$ be a free resolution of k as an $\mathcal{H}$-module. Then the groups $\mathrm{Ext}^n_{\mathcal{H}}(k,k)$ are the cohomology groups of the cochain complex

$$\mathrm{Hom}_{\mathcal{H}}(P_0,k) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(P_1,k) \longrightarrow \cdots.$$

Example 4.2. Consider the Hopf algebra $\mathcal{H} = E_k(x)$. Then k has a "periodic" free resolution

$$\cdots \longrightarrow E(x) \xrightarrow{x} E(x) \xrightarrow{x} E(x) \xrightarrow{x} E(x) \xrightarrow{\varepsilon} k.$$

This is best represented graphically:



The vector space $\mathrm{Hom}_{E(x)}(E(x),k)$ is one-dimensional, with basis element ε. Thus the chain complex $\mathrm{Hom}_{E(x)}(P_*,k)$ becomes

$$k\{\varepsilon\} \xrightarrow{0} k\{\varepsilon\} \xrightarrow{0} k\{\varepsilon\} \xrightarrow{0} \cdots.$$

Thus $\mathrm{H}^n(E(x)) \cong k$ for all $n \geq 0$.

Fri, Mar. 24

Example 4.3. Consider the Hopf algebra $\mathbb{F}_2[C_2]$. Again, we can write down a periodic free resolution

$$\cdots \mathbb{F}_2[C_2] \xrightarrow{g-1} \mathbb{F}_2[C_2] \xrightarrow{N} \mathbb{F}_2[C_2] \xrightarrow{g-1} \mathbb{F}_2[C_2] \xrightarrow{\varepsilon} \mathbb{F}_2.$$

Here we write g for the generator of C_2 and $N = 1 + g$. It follows that $\mathrm{H}^n(\mathbb{F}_2[C_2]) \cong \mathbb{F}_2$ for all $n \geq 0$.

Mon, Mar. 27

Note that if $\mathcal{H}$ is a *graded* Hopf algebra, as in many of our recent examples, then Ext inherits a second grading. For instance, in the exterior example $E(x)$, if x is placed in degree n, then our resolution really looks like

$$\cdots E(x)[3n] \xrightarrow{x} E(x)[2n] \xrightarrow{x} E(x)[n] \xrightarrow{x} E(x) \xrightarrow{\varepsilon} k.$$

Here the notation $M^*[n]$ means just shifting the degrees on the graded module M^* by n. Applying $\mathrm{Hom}(-, k)$ gives the complex

$$k \xrightarrow{0} k[2] \xrightarrow{0} k[4] \xrightarrow{0} \cdots$$

Thus $\mathrm{Ext}^s_{E(x_n)}(k,k) \cong k[ns]$. Alternatively, we write $\mathrm{Ext}^{s,t}(k,k)$, where s refers to the (co)homological degree, and t refers to the 'internal' degree. So we here get

$$\mathrm{Ext}^{s,t}_{E(x_n)}(k,k) \cong \begin{cases} k & t = n \cdot s \\ 0 & \text{else.} \end{cases}$$

We have written down resolutions in specific examples. In fact, there is a canonical resolution that exists quite generally.

Definition 4.4. Let A be an augmented algebra over k (for example, a Hopf algebra). Then the **bar resolution** of k is $E_n A := A^{\otimes n+1}$, for $n \geq 0$, with differentials

$$\cdots \longrightarrow A^{\otimes 3} \xrightarrow{d_2} A^{\otimes 2} \xrightarrow{d_1} A \xrightarrow{\varepsilon} k$$

given by

$$d_n(a_0|\cdots|a_n) = a_0a_1|a_2|\cdots|a_n - a_0|a_1a_2|\cdots|a_n + \cdots + (-1)^{n-1}a_0|a_1|\cdots|a_{n-1}a_n + (-1)^n\varepsilon(a_n)a_0|a_1|\cdots|a_{n-1}.$$

For example,

$$d_1(a|b) = ab - \varepsilon(b)a, \qquad d_2(a|b|c) = ab|c - a|bc + \varepsilon(c)a|b.$$

Proposition 4.5. *The bar resolution is a resolution of k.*

Proof. We show that the chain complex E_*A is in fact chain-homotopy equivalent to k. Since k is a retract of A via $k \xrightarrow{\eta} A = E_0A \xrightarrow{\varepsilon} k$, it suffices to show that the identity map of E_*A is homotopic to $E_*A \xrightarrow{\varepsilon} k \xrightarrow{\eta} E_*A$. Define $h_n : E_nA \longrightarrow E_{n+1}A$ by $h_n(a_0|\cdots|a_n) = 1|a_0|\cdots|a_n$. We claim that this is the desired chain homotopy. For example,

$$\begin{aligned} d_1h_0(a) &= d_1(1|a) = a - \varepsilon(a) = (\mathrm{id} - \eta\varepsilon)(a), \\ (d_2h_1 + h_0d_1)(a|b) &= d_2(1|a|b) + h_0(ab - \varepsilon(b)a) \\ &= a|b - 1|ab + \varepsilon(b)1|a + 1|ab - \varepsilon(b)1|a = a|b \end{aligned}$$

$$\begin{tikzcd} A^{\otimes 3} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "0"'] \arrow[d, "d_2"'] & A^{\otimes 3} \arrow[d, "d_2"] \\ A^{\otimes 2} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "0"'] \arrow[d, "d_1"'] \arrow[ur, "h_1"] & A^{\otimes 2} \arrow[d, "d_1"] \\ A \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\eta\varepsilon"'] \arrow[ur, "h_0"] & A \end{tikzcd}$$

∎

Wed, Mar. 29

It follows that the bar resolution may be used to compute $\mathrm{Ext}_{\mathcal{H}}(k,k)$. We call the resulting complex $\mathrm{Hom}_A(E_*A,k)$, obtained by dualizing the bar resolution, the **cobar complex**. It takes the form $coB^*_A(k,k) =$

$$\begin{aligned} & \mathrm{Hom}_A(A,k) \xrightarrow{d^0} \mathrm{Hom}_A(A^{\otimes 2},k) \xrightarrow{d^1} \mathrm{Hom}_A(A^{\otimes 3},k) \xrightarrow{d^2} \ldots \\ \cong \quad & k \xrightarrow{d^0} \mathrm{Hom}_k(A,k) \xrightarrow{d^1} \mathrm{Hom}_k(A^{\otimes 2},k) \xrightarrow{d^2} \ldots \\ \cong \quad & k \xrightarrow{d^0} A^\vee \xrightarrow{d^1} (A^\vee)^{\otimes 2} \xrightarrow{d^2} \ldots \end{aligned}$$

Corollary 4.6. *The cohomology of the augmented algebra A is computed by the cobar complex:*

$$\mathrm{H}^n(A) \cong \mathrm{H}^n(coB^*_A(k,k)).$$

Unfortunately, the cobar complex is in general much too large therefore impractical for computations. One minor improvement can be obtained as follows.

Definition 4.7. Define the subspace $D_nA \subseteq E_nA$ by

$$D_nA = A \otimes k \otimes A^{\otimes n-1} + A \otimes A \otimes k \otimes A^{\otimes n-2} + A \otimes A^{\otimes 2} \otimes k \otimes A^{\otimes n-3} + \cdots + A \otimes A^{\otimes n-1} \otimes k.$$

By convention, $D_0A = 0$.

Proposition 4.8. *The D_nA's define a subcomplex of E_nA, and this complex is chain-homotopy equivalent to 0.*

Definition 4.9. The **reduced (or normalized) bar resolution** of k is $\overline{E}_*A = E_*A/D_*A$.

Recalling that the cokernel $k \xrightarrow{\eta} A$ is isomorphic to I_A, it follows that $\overline{E}_nA = A \otimes I_A^{\otimes n}$. The differential on $\overline{E}_*A$ is the same as for E_*A, except that the last term, which involves $\varepsilon(a_n)$, is now eliminated. When we pass the reduced bar resolution into $\mathrm{Hom}_A(-,k)$, we get the **reduced cobar complex** $\overline{coB}_A(k,k)$:

$$k \xrightarrow{0} I_A^\vee \xrightarrow{d^1} (I_A^\vee)^{\otimes 2} \xrightarrow{d^2} \ldots$$

Actually the differentials become quite nice: $d^1 = \overline{\Delta} : I^\vee \longrightarrow (I^\vee)^{\otimes 2}$ and $d^2 : (I^\vee)^{\otimes 2} \longrightarrow (I^\vee)^{\otimes 3}$ is $\mathrm{id} \otimes \overline{\Delta} + \overline{\Delta} \otimes \mathrm{id}$.

Proposition 4.10. *We have an identification* $\mathrm{H}^1(A) \cong (Q_A)^\vee \cong P(A^\vee)$.

Proof. Since $d^0 = 0$, it suffices to determine the d^1-cocycles. These are precisely the primitives, which according to Proposition 3.27 are dual to Q_A. ∎

Example 4.11. Recall that $\mathcal{E}(n) \subseteq \mathcal{A}$ was exterior on classes $Q_0, \ldots, Q_n$, where $\deg Q_i = 2^{i+1} - 1$. It follows that

$$\mathrm{H}^1(\mathcal{E}(n)) \cong \mathbb{F}_2\{v_0, v_1, \ldots, v_n\}, \qquad v_i \in \mathrm{Ext}^{1,2^{i+1}-1}_{\mathcal{E}(n)}(\mathbb{F}_2, \mathbb{F}_2).$$

Example 4.12. Recall that $\mathcal{A}(n) \subseteq \mathcal{A}$ was the subalgebra generated by $\boxed{1}, \ldots, \boxed{2^n}$. It follows that

$$\mathrm{H}^1(\mathcal{A}(n)) \cong \mathbb{F}_2\{h_0, h_1, \ldots, h_n\}, \qquad h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}(n)}(\mathbb{F}_2, \mathbb{F}_2).$$

Similarly,

$$\mathrm{H}^1(\mathcal{A}) \cong \mathbb{F}_2\{h_0, h_1, \ldots\}.$$

4.1. **Products in** Ext**.** There are several ways to think about the *ring* structure on Ext.

Multiplying in the cobar complex:

As discussed above, the reduced cobar complex is given in degree n by $\overline{coB}^n_A(k,k) \cong (I_A^\vee)^{\otimes n}$. There is then an obvious multiplication induced by the identification

$$\overline{coB}^n_A(k,k) \otimes \overline{coB}^i_A(k,k) \cong (I_A^\vee)^{\otimes n} \otimes (I_A^\vee)^{\otimes i} \cong (I_A^\vee)^{\otimes (n+i)} \cong \overline{coB}^{n+i}_A(k,k).$$

This simply looks like

$$a_1|\cdots|a_n \cdot b_1|\cdots b_i := a_1|\cdots|a_n|b_1|\cdots b_i.$$

Example 4.13. In $E(x)$, let us write $y = x^\vee$, so that $(E(x))^\vee \cong E(y)$. Then the class v is represented by $|y|$, and

$$v^n = (|y|)^n = |y|y|\cdots|y|.$$

Mon, Apr. 3

Example 4.14. Recall that $\mathcal{E}(1) \cong E(Q_0, Q_1)$, where Q_0 and Q_1 are primitive and live in degree 1 and 3, respectively. The dual is $\mathcal{E}(1)_* \cong E(z_1, z_2)$, with z_1 and z_2 primitive and in degrees 1 and 3. Thus z_1 and z_2 give rise to classes in Ext^1. Let

$$v_0 := [z_1], \qquad v_1 := [z_2].$$

Note that

$$d^1([z_1 z_2]) = \overline{\Delta}(z_1 z_2) = z_1|z_2 + z_2|z_1 = v_0 v_1 + v_1 v_0,$$

so that v_0 and v_1 commute with each other. By inspecting $d^2 : (I^\vee)^{\otimes 2} \longrightarrow (I^\vee)^{\otimes 3}$, it can be seen that $\mathrm{Ext}^2 \cong \mathbb{F}_2\{v_0^2, v_1^2, v_0 v_1\}$. In fact, $\mathrm{Ext}_{\mathcal{E}(1)} \cong \mathbb{F}_2[v_0, v_1]$.

The examples we have seen so far have all given us commutative rings as Ext-algebras.

Proposition 4.15. *(cf [McC, Theorem 9.9]) If $\mathcal{H}$ is a cocommutative Hopf algebra over k, then $\mathrm{Ext}_{\mathcal{H}}(k,k)$ is a graded-commutative ring, with signs*

$$\alpha \cdot \beta = (-1)^{ss' + tt'} \beta \cdot \alpha$$

if $\alpha \in \mathrm{Ext}^{s,t}$ and $\beta \in \mathrm{Ext}^{s',t'}$.

Extensions: Recall that if A is a ring and M and N are left modules, there is an isomorphism of $\mathrm{Ext}^1(M,N)$ with the set of (isomorphism classes of) 'extensions' of M by N, meaning short exact sequences

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Given $\alpha \in \mathrm{Ext}^1_A(M,N)$, represent this as a map $F_1 \longrightarrow N$, where $F_* \longrightarrow M$ is a resolution. Then define $E_\alpha := N \oplus_{F_1} F_0$ as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1/F_2 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E_\alpha \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Proposition 4.16. *The set $\mathrm{Ext}^n(M,N)$ can be identified with (equivalence classes) of length n extensions*

$$0 \longrightarrow N \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \ldots \longrightarrow E_1 \longrightarrow M \longrightarrow 0.$$

Here the equivalence relation is the one generated by maps of extensions.

If $n \geq 2$, the element $0 \in \mathrm{Ext}^n(M,N)$ corresponds to the 'trivial' length n extension

$$0 \longrightarrow N \longrightarrow N \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow M \longrightarrow M \longrightarrow 0.$$

Unfortunately, the (additive) group structure is not so easy to describe on the extension side. But the multiplication is very nice: given $\alpha \in \mathrm{Ext}^n(M', M'')$ and $\beta \in \mathrm{Ext}^k(M, M')$, represented as extensions

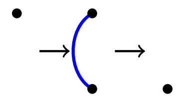
$$0 \longrightarrow M'' \longrightarrow E_n \longrightarrow \ldots \longrightarrow E_1 \longrightarrow M' \longrightarrow 0, \qquad 0 \longrightarrow M' \longrightarrow F_k \longrightarrow \ldots \longrightarrow F_1 \longrightarrow M \longrightarrow 0,$$

we simply splice them together to get the extension

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r] & E_n \arrow[r] & \ldots \arrow[r] & E_1 \arrow[rr, dashed] \arrow[dr] & & F_k \arrow[r] & \ldots \arrow[r] & F_1 \arrow[r] & M \arrow[r] & 0, \\
& & & & & M' \arrow[ur] & & & & &
\end{tikzcd}$$

in $\mathrm{Ext}^{n+k}(M, M'')$.

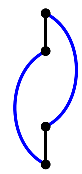
Example 4.17. For $A = E(x)$, the element in $\mathrm{Ext}^1_{E(x)}(k,k)$ can be viewed as the nontrivial extension

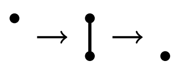
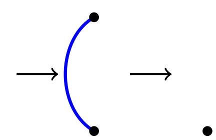


Splicing n of these together gives the nontrivial extension of length n. This shows that

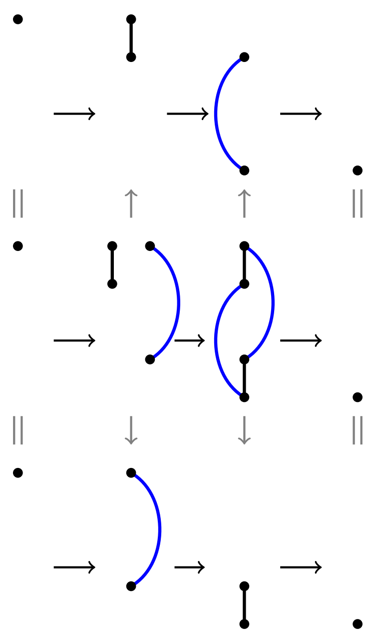
$$\mathrm{Ext}^{*,*}_{E(x_n)}(k,k) \cong k[v], \qquad |v| = (1,n).$$

Wed, Apr. 5

Example 4.18. We may represent $\mathcal{E}(1) \cong E(Q_0, Q_1)$ pictorially as  . The vertical black lines denote Q_0-multiplication, and the curved blue arrows denote Q_1-multiplication. Then $v_0 \in \mathrm{Ext}^{1,1}_{\mathcal{E}(1)}$ and $v_1 \in \mathrm{Ext}^{1,3}_{\mathcal{E}(1)}$ are given by the extensions

v_0 :  and v_1 : 

The equality $v_0 \cdot v_1 = v_1 \cdot v_0$ can be represented as the following maps of extensions:



As we have seen from examples, not all resolutions are equally convenient for computing Ext. The bar resolution is quite large, while in some cases we can make smaller (sometimes periodic) resolutions. A convenient class of resolutions is as follows.

Definition 4.19. If A is an augmented algebra and M is an A-module, then a **minimal resolution** of M is a resolution $P_* \longrightarrow M$ such that the image of each differential satisfies $d_n(P_n) \subseteq I_A \cdot P_{n-1}$.

The reason for the terminology is that if a resolution is *not* minimal, then up to a change of basis, there must be an instance in which d_n carries a module generator of P_n to a module generator of P_{n-1}. But then those summands can be removed from the resolution, creating a smaller resolution.

Proposition 4.20. *If $P_* \longrightarrow M$ is a minimal resolution, then the cochain complex $\mathrm{Hom}_A(P_*, k)$ has zero differentials.*

Proof. Since $I_A \cdot k = 0$, any element $f \in \mathrm{Hom}_A(P_n, k)$ must vanish on $I_A \cdot P_n$. But now $d^n(f) = f \circ d_{n+1} = 0$ since $\mathrm{im}(d_{n+1}) \subseteq I_A \cdot P_n$. ∎

Corollary 4.21. *If $P_* \longrightarrow M$ is a minimal free resolution of M, then $\mathrm{Ext}^n_A(M,k) \cong \mathrm{Hom}_A(P_n, k)$.*

Example 4.22. Our periodic resolution of $E(x)$ was a minimal free resolution.

Mon, Apr. 10

4.2. Spectral sequences.

We have discussed some ways to calculate $\mathrm{Ext}_{\mathcal{H}}$. Many times, a head-on approach is not practical. In general, we would like to reduce an intricate calculation to a number of simpler ones.

For instance, suppose we have a cochain complex C^* and a filtration by subcomplexes

$$C^* = F_0C^* \supset F_1C^* \supset F_2C^* \supset \cdots .$$

Then each quotient $F_nC^*/F_{n+1}C^*$ inherits a differential from C^*, and we can consider $E_0 := \bigoplus_{n\geq 0} \mathrm{H}^*(F_nC^*/F_{n+1}C^*)$. Of course, this is in general quite different from $\mathrm{H}^*(C^*)$, but we can hope that this is at least a step in the direction of computing the latter.

One general example of a filtered complex comes from a **double complex**:

$$\begin{tikzcd}
C^{0,2} \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[r, "d_h^{1,2}"] & C^{2,2} \\
C^{0,1} \arrow[r, "d_h^{0,1}"] \arrow[u, "d_v^{0,1}"] & C^{1,1} \arrow[r, "d_h^{1,1}"] \arrow[u, "d_v^{1,1}"] & C^{2,1} \arrow[u, "d_v^{2,1}"] \\
C^{0,0} \arrow[r, "d_h^{0,0}"] \arrow[u, "d_v^{0,0}"] & C^{1,0} \arrow[r, "d_h^{1,0}"] \arrow[u, "d_v^{1,0}"] & C^{2,0} \arrow[u, "d_v^{2,0}"]
\end{tikzcd}$$

From a double complex we can form the *total complex*, $\mathrm{Tot}^n(C^{*,*}) := \bigoplus_{i+j=n} C^{i,j}$ with $d^n : \mathrm{Tot}^n \longrightarrow \mathrm{Tot}^{n+1}$ defined by $d^n = \bigoplus_{i+j=n}(-1)^i(d_h^{i,j} + d_v^{i,j})$. We now define a filtration on $\mathrm{Tot}^*(C^{*,*})$ by letting $F_n \mathrm{Tot}^*(C^{*,*})$ be the subcomplex consisting only of columns $C^{\geq n,*}$.

Then each $F_n \mathrm{Tot}^* / F_{n+1} \mathrm{Tot}^*$ is just the column $C^{n,*}$. So, writing $d_0 = d_v$ we let

$$E_0 = \bigoplus_n (F_n \mathrm{Tot}^* / F_{n+1} \mathrm{Tot}^*, d_0), \qquad E_1 := \mathrm{H}^*(E_0, d_0).$$

Now on E_1 we still have the horizontal differential left over, so we let d_1 be induced by d_h, and we define

$$E_2 := \mathrm{H}^*(E_1, d_1).$$

But now it turns out we can define a further differential, which we call d_2, as follows. Suppose that $\alpha \in E_1$ is a d_1-cocycle and so survives to E_2. Let $x \in \mathrm{Tot}$ be a choice of representative for α. To say that α is a d_1-cocycle means that $d(x)$ must be a d_0-boundary. Thus we have

$$\begin{tikzcd}
x \arrow[r, mapsto] & y \\
& z \arrow[u, mapsto] \arrow[r, mapsto, dashed] & w
\end{tikzcd}$$

Now we define $d_2(\alpha) := [w]$. We then define $E_3 := \mathrm{H}^*(E_2, d_2)$, etc.

Example 4.23. Consider the double complex

$$\begin{tikzcd}
u \\
x \arrow[u, mapsto] \arrow[r, mapsto] & v \\
& y \arrow[u, mapsto] \arrow[r, mapsto] & w
\end{tikzcd}$$

Note that $\mathrm{Tot}^1 \cong k\{x, y\}$ and $\mathrm{Tot}^2 \cong k\{u, v, w\}$. The differential $d : \mathrm{Tot}^1 \hookrightarrow \mathrm{Tot}^2$ is injective, and the cohomology is one-dimensional, concentrated in degree 2. Using the above approach, we have that E_0 is

$$E_0 : \quad \begin{tikzcd}
u \\
x \arrow[u, mapsto] & v \\
& y \arrow[u, mapsto] & w,
\end{tikzcd}$$

so that E_1 is

$$E_1 : \quad \begin{tikzcd}
0 \\
0 & 0 \\
& 0 & w.
\end{tikzcd}$$

Thus we have $E_1 \cong \mathrm{H}^*(\mathrm{Tot}^*)$.

Example 4.24. Consider the "double" complex

$$x \longmapsto y.$$

Then E_0 is

$$E_0 : \qquad x \qquad y,$$

while E_1 is

$$E_1 : \qquad x \longmapsto y.$$

Thus E_2 and all later terms vanish, and we recover $\mathrm{H}^*(\mathrm{Tot}^*) = 0$.

Example 4.25. Consider the double complex

$$\begin{tikzcd}
x \arrow[r, mapsto] & v \\
& y \arrow[u, mapsto] \arrow[r, mapsto] & w
\end{tikzcd}$$

In this case, we see that $\mathrm{H}^*(\mathrm{Tot}^*) = 0$. Now E_0 is

$$\begin{tikzcd} E_0 : & x & v & \\ & & y \arrow[u] & w, \end{tikzcd}$$

so that E_1 is

$$\begin{tikzcd} E_1 : & x & 0 & \\ & & 0 & w. \end{tikzcd}$$

There simply is no room for a d_1-differential here. However, when we consider the same picture as the E_2-page, we have a differential

$$\begin{tikzcd} E_2 : & x \arrow[drr] & 0 & \\ & & 0 & w. \end{tikzcd}$$

Then E_3 and all later terms vanish.

This structure we have discussed, namely the collection of complexes (E_r, d_r), is known as a **spectral sequence**. In this case, the spectral sequence is supposed to eventually tell us about $\mathrm{H}^*(\mathrm{Tot}^* C)$, and we write

$$E_1 = \mathrm{H}^*(\bigoplus_n F_nC/F_{n+1}C) \Rightarrow \mathrm{H}^*(\mathrm{Tot}\, C).$$

$\boxed{\textbf{Wed, Apr. 12}}$

One spectral sequence arises in the context of extensions of Hopf algebras.

Definition 4.26. A sequence of Hopf algebra maps

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

is called an **extension of Hopf algebras** if (1) f is an inclusion of a normal Hopf algebra (see 2.45) and (2) the map g identifies $\mathcal{C}$ with the quotient $B/\!/A = B/I_A \cdot B$. It is a **central extension** if $\mathcal{A}$ is central in $\mathcal{B}$, meaning that $f(a)b = bf(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Theorem 4.27. *Suppose $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ is a central extension of Hopf algebras. Then there is a spectral sequence, known as the Cartan-Eilenberg spectral sequence, with*

$$E_2 \cong \mathrm{H}^*(\mathcal{A}) \otimes \mathrm{H}^*(\mathcal{C}) \Rightarrow \mathrm{H}^*(\mathcal{B}).$$

In the E_2-term of the spectral sequence, $\mathrm{H}^*(\mathcal{A})$ appears in the column $E_2^{0,*}$, whereas $\mathrm{H}^*(\mathcal{C})$ appears as the row $E_2^{*,0}$.

Example 4.28. Consider the central extension

$$E(Q_0) \hookrightarrow \mathcal{E}(1) \longrightarrow E(Q_1).$$

This is in fact a split extension, meaning that $\mathcal{E}(1) \cong E(Q_0) \otimes E(Q_1)$, though we will not use this. Recall that $\mathrm{H}^*(E(Q_0)) \cong \mathbb{F}_2[v_0]$ and $\mathrm{H}^*(E(Q_1)) \cong \mathbb{F}_2[v_1]$, with $|v_0| = (1,1)$ and $|v_1| = (1,3)$. Thus the E_2-page looks like

$$E_2\colon \quad \begin{array}{|c|c|c|c|} \hline v_0^3 & v_0^3v_1 & v_0^3v_1^2 & v_0^3v_1^3 \\ \hline v_0^2 & v_0^2v_1 & v_0^2v_1^2 & v_0^2v_1^3 \\ \hline v_0 & v_0v_1 & v_0v_1^2 & v_0v_1^3 \\ \hline 1 & v_1 & v_1^2 & v_1^3 \\ \hline \end{array}$$

Now there is a potential differential $d_2(v_0) \overset{?}{=} v_1^2$, but this cannot occur since v_0 has internal degree 1, whereas v_1^2 has internal degree 6.

But now we exploit the fact that this is a multiplicative spectral sequence: E_2 is a ring, and the differential d_2 satifies the Leibniz rule. In particular, once we know that the generators v_0 and v_1 are both cycles, it follows that every class is a cycle. Thus d_2 vanishes. If we consider d_3 or higher, then both v_1 and v_0 are necessarily cycles just for degree reasons (the d_r would land outside the grid), so v_0 and v_1 are **permanent cycles**. Again, since these are algebra generators, it follows that every class is a permanent cycle, so that $E_2 = E_\infty$ (we say the spectral sequence **collapses** at E_2). We conclude that $\mathrm{H}^*(\mathcal{E}(1)) \cong \mathbb{F}_2[v_0, v_1]$ as previously stated.

Example 4.29. Next we consider $\mathcal{A}(1)$. We can express this as a central extension

$$E(Q_1) \hookrightarrow \mathcal{A}(1) \longrightarrow \mathcal{A}(1)/Q_1.$$

We know $\mathrm{H}^*(E(Q_1)) \cong \mathbb{F}_2[v_1]$, but we have not yet computed $\mathrm{H}^*(\mathcal{A}(1)/Q_1)$, so we begin by computing this.

We have a central extension

$$\mathcal{A}(0) = E(\boxed{1}) \hookrightarrow \mathcal{A}(1)/Q_1 \longrightarrow E(\boxed{2}).$$

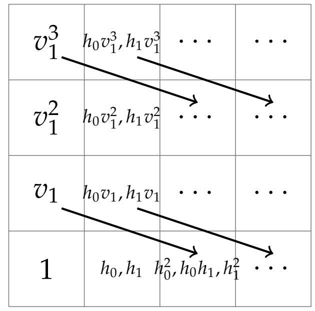
Now $\mathrm{H}^*(E(\boxed{1})) \cong \mathbb{F}_2[h_0]$ and $\mathrm{H}^*(E(\boxed{2})) \cong \mathbb{F}_2[h_1]$, with $|h_0| = (1,1)$ and $|h_1| = (1,2)$. Applying the Cartan-Eilenberg spectral sequence to $\mathcal{A}(1)/Q_1$ gives the E_2-page

$$E_2\colon \quad \begin{array}{|c|c|c|c|} \hline h_0^3 & h_0^3h_1 & h_0^3h_1^2 & h_0^3h_1^3 \\ \hline h_0^2 & h_0^2h_1 & h_0^2h_1^2 & h_0^2h_1^3 \\ \hline h_0 & h_0h_1 & h_0h_1^2 & h_0h_1^3 \\ \hline 1 & h_1 & h_1^2 & h_1^3 \\ \hline \end{array}$$

By the same reasoning as in the previous example, the spectral sequence collapses at E_2, so that $\mathrm{H}^*(\mathcal{A}(1)/Q_1) \cong \mathbb{F}_2[h_0, h_1]$.

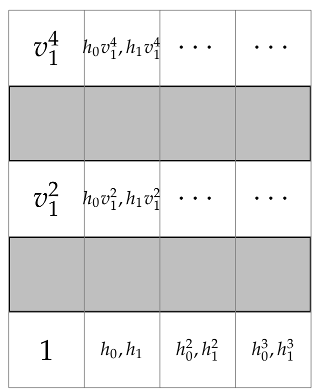
We now return to $\mathcal{A}(1)$. The E_2-page of the spectral sequence here takes the form

$E_2:$

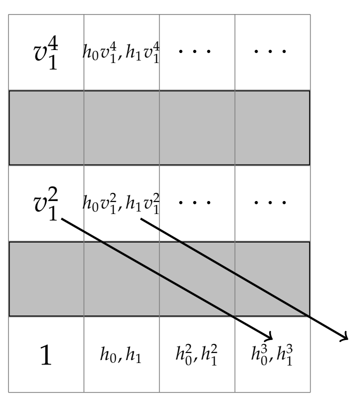


In this case, we do have a differential $d_2(v_1) = h_0h_1$ (note that both have internal degree 3). One way to see this is to recall that $\mathrm{H}^1(\mathcal{A}(1))$ corresponds to the space of indecomposables of $\mathcal{A}(1)$, which are $\boxed{1}$ and $\boxed{2}$. These correspond to h_0 and h_1, so v_1 cannot survive the spectral sequence. The only way this can happen is if it supports a d_2, and the stated one is the only possible one from degree considerations. Since we are working mod 2, it follows that v_1^2 is a cycle, but v_1^3 supports a d_2, hitting $h_0h_1v_1^2$. Now the E_3-term is given by

$E_3:$



$E_3:$



By building a minimal free resolution of $\mathbb{F}_2$ over $\mathcal{A}(1)$, you can show that v_1^2 does not survive the spectral sequence. This forces v_1^2 to support a d_3. For degree reasons, that must be $d_3(v_1^2) = h_1^3$. Note that this means $d_3(h_1v_1^2) = h_1^4$, but $h_0v_1^2$ is a cycle since $h_0h_1^3$ was already 0 in E_3. Thus the E_4

is given by

E_4:

v_1^4	$h_0v_1^4, h_1v_1^4$	$\cdots$	$\cdots$
	$h_0v_1^2$	$h_0^2v_1^2$	$\cdots$
1	h_0, h_1	h_0^2, h_1^2	h_0^3

The E_4 page is generated by h_0, h_1, $h_0v_1^2$, and v_1^4. The first three generators must be permanent cycles. The class v_1^4 cannot support a d_4, (this would land in the zero row $E^{*,1}$), but it could support a d_5. However, $d_5(v_1^4) \in E_5^{5,0} \cong \mathbb{F}_2\{h_0^5\}$. Since h_1^4 has internal degree 12 and h_0^5 has degree 5, we conclude that $d_5(v_1^4) = 0$, so that v_1^4 is also a permanent cycle, and the spectral sequence degenerates at E_4. Thus, writing $a := h_0v_1^2$ and $b := v_1^4$,

$$\mathrm{H}^*(\mathcal{A}(1)) \cong \mathbb{F}_2[h_0, h_1, a, b]/(h_0h_1, h_1^3, h_1a, a^2 + h_0^2b).$$

Mon, Apr 17

Last week, we discussed the Cartan-Eilenberg and saw how to use this to compute $\mathrm{Ext}_{\mathcal{A}(1)}$. To apply these ideas to $\mathrm{Ext}_{\mathcal{A}}$, we would need to express $\mathcal{A}$ as an extension of Hopf algebras. One way to do this is the extension

$$\mathcal{E} \hookrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\!/\mathcal{E}.$$

However, $\mathcal{E} \hookrightarrow \mathcal{A}$ is not central, making the E_2-term more complicated. The quotient $\mathcal{A}/\!/\mathcal{E}$, it turns out, is a *doubled* copy of $\mathcal{A}$, meaning that there is a degree-doubling isomorphism $\mathcal{A} \cong \mathcal{A}/\!/\mathcal{E}$. So we could not use the Cartan-Eilenberg spectral sequence and this extension to compute $\mathrm{Ext}_{\mathcal{A}}$, since that is one of the ingredients for the E_2-term of the spectral sequence.

4.3. **The May spectral sequence.** Another spectral sequence was developed by Peter May in his thesis. This arises from filtering $\mathcal{A}$ by the augmentation ideal I.

Theorem 4.30 (May)**.** *There is a spectral sequence*

$$E_2 = \mathrm{Ext}_{\mathrm{gr}_I^* \mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \mathrm{gr}^* \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

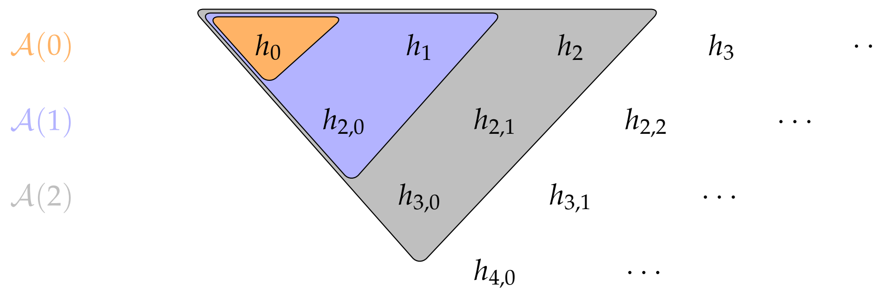
Moreover, $\mathrm{Ext}_{\mathrm{gr}_I^* \mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{H}^*(\mathbb{F}_2[h_{i,j}]_{i \geq 1,\, j \geq 0}, d)$*, where*

$$d(h_{i,j}) = \sum_{0<k<i} h_{k,j} h_{i-k,k+j}$$

and $h_{i,j}$ has degree $(1, 2^j \cdot (2^i - 1))$.

Usually the elements $h_{1,j}$ are simply written as h_j. It turns out that all differentials in this spectral sequence occur on even pages. The elements $h_{i,j}$ come from the elements $z_i^{2^j} \in \mathcal{A}_*$. In particular, if we instead consider $\mathcal{A}(n)$, there is a similar description, where we only include the $h_{i,j}$ with

$i + j \leq n + 1$.



Example 4.31. Consider the case of $\mathcal{A}(0) = E(\boxed{1})$. Then we only include $h_{1,0} = h_0$. Here, we have $E_2 \cong \mathbb{F}_2[h_0]$, which we already know is the correct answer for $\mathrm{Ext}_{\mathcal{A}(0)}$.

Example 4.32. Consider $\mathcal{A}(1)$. Then we include $h_{1,0} = h_0$, $h_{1,1} = h_1$, and $h_{2,0}$, with $d(h_{2,0}) = h_0 h_1$. This is the differential we discussed last week, and we get that the May E_2-page is $\mathbb{F}_2[h_0, h_1, h_{2,0}^2]/h_0 h_1$, which was the E_3-page in the Cartan-Eilenberg spectral sequence.

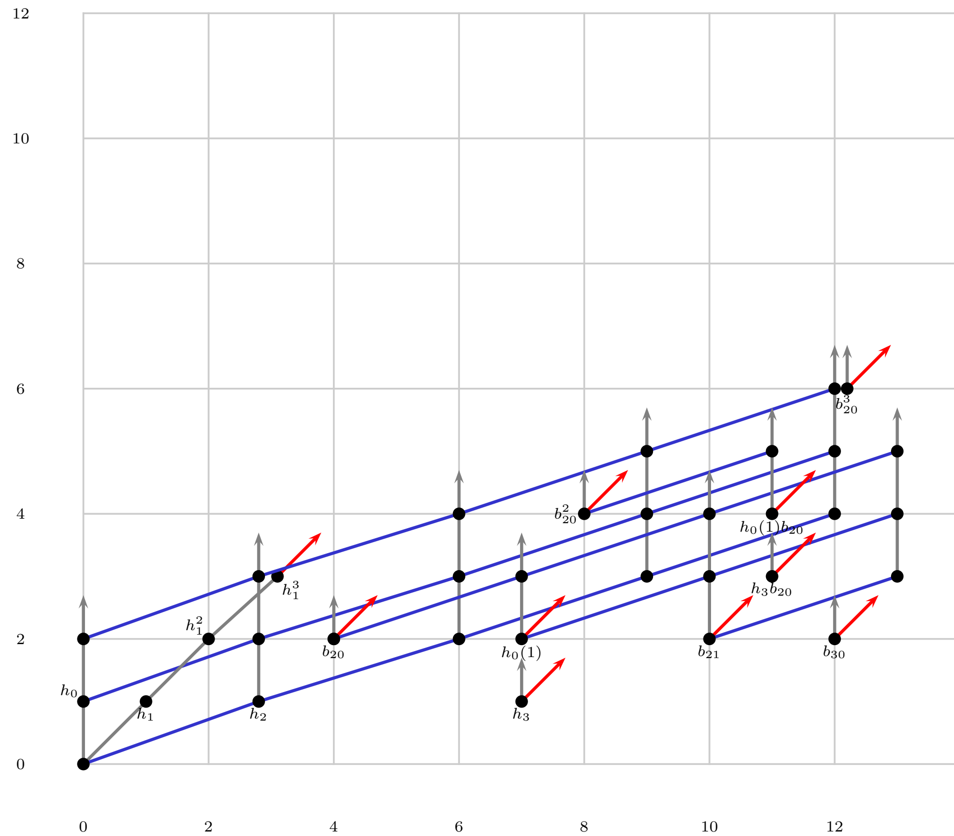
Wed, Apr. 19

For reasons we will discuss next week, topologists usually like to depict $\mathrm{Ext}^{s,t}$ using the Adams grading, meaning that the horizontal axis corresponds to $t - s$, while the vertical axis is s. In this convention, May differentials always go up one and to the left one (the May filtration m is not depicted in these charts).

Here is a table of multiplicative generators for the May E_2-term, in the range $t - s \leq 13$. The element b_{ij} is represented in the E_0 page by h_{ij}^2. The element $h_0(1)$ is $h_{20}h_{21} + h_1 h_{30}$.

generator	$t-s$	s	m	d_2
h_0	0	1	1	
h_1	1	1	1	
h_2	3	1	1	
h_3	7	1	1	
b_{20}	4	2	4	$h_0^2 h_2 + h_1^3$
b_{21}	10	2	4	$h_1^2 h_3 + h_2^3$
b_{30}	12	2	6	$h_1 b_{21} + h_3 b_{20}$
$h_0(1)$	7	2	4	$h_0 h_2^2$

May E_2-page, $t - s \leq 13$:

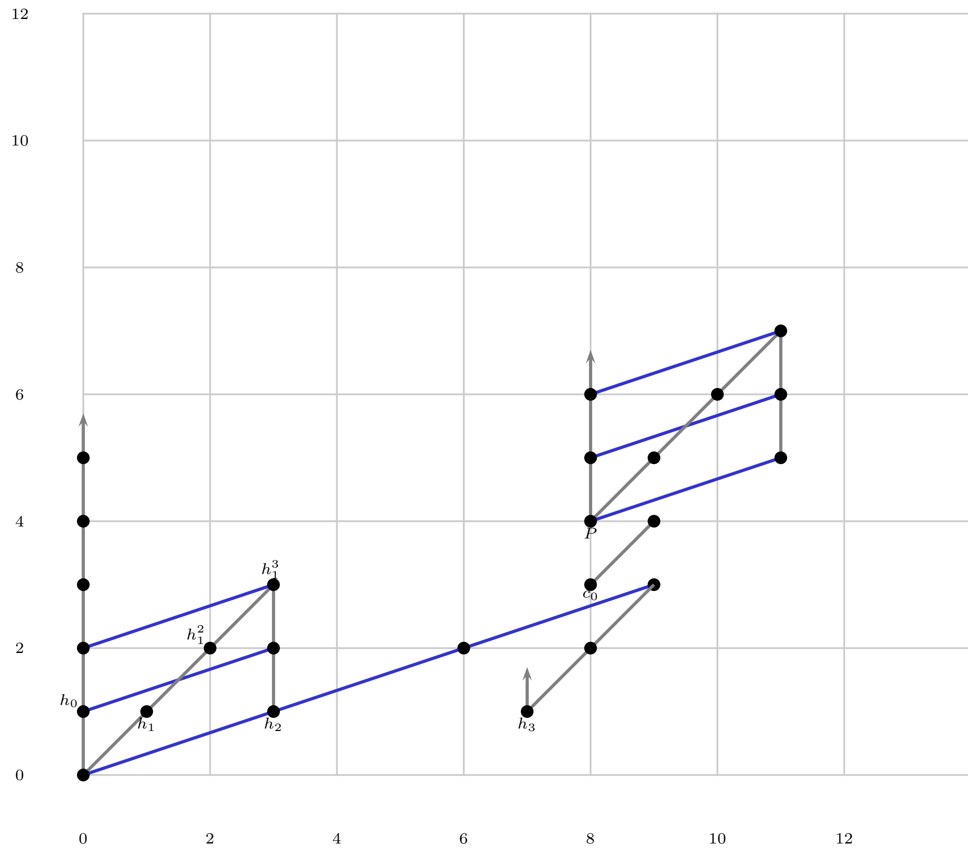


Represented in the figure are a number of relations:

$$
\begin{aligned}
h_0h_1 &= 0, \\
h_1h_2 &= 0, \\
h_2h_3 &= 0, \\
h_2b_{20} &= h_0h_0(1), \\
h_2h_0(1) &= h_0b_{21}.
\end{aligned}
$$

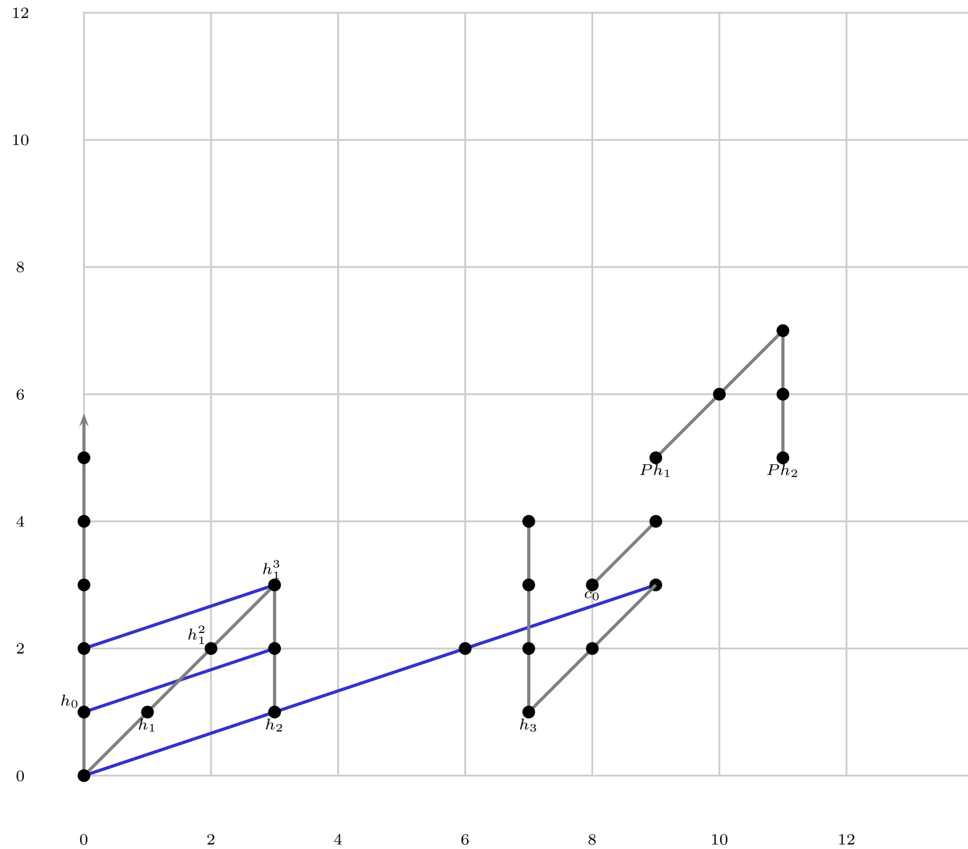
The d_2-differentials listed in the above table lead to the May E_4-page displayed below, where $P = b_{20}^2$ and $c_0 = h_1h_0(1)$.

May E_4-page, $t-s \leq 13$:



There is one more differential, $d_4(P) = h_0^4 h_3$. This yields the picture for $\mathrm{Ext}_{\mathcal{A}}$ in the range $t-s \leq 13$.

May E_∞-page, $t-s \le 13$:



One result that is useful for deducing differentials in the May spectral sequence is

Theorem 4.33 (Adams vanishing line)**.** *We have*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t} = 0, \qquad \text{if } s > \frac{1}{2}(t-s) + \frac{3}{2} \text{ and } t-s > 0$$

The main thing to remember is that $\mathrm{Ext}_{\mathcal{A}}$ vanishes above a line of slope $\frac{1}{2}$ (in positive stems).

Wed, Apr 26

4.4. **The Adams spectral sequence.**

We have spent quite a bit of time now on computing Ext of $\mathcal{A}$ and some of its sub-Hopf algebras. But why? These are interesting computations, and they have allowed us to introduce some interesting tools, but there are more compelling reasons to care about these.

Theorem 4.34 (Adams)**.** *There is a (Adams) spectral sequence with*

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}^{stable}(S^0)_2^\wedge.$$

The target groups are the *stable homotopy groups* of spheres. Recall that the stable homotopy group $\pi_n^{stable}(S^0)$ is defined to be the limiting, or stable, value in the sequence

$$\pi_n(S^0) \xrightarrow{\Sigma} \pi_{n+1}(S^1) \xrightarrow{\Sigma} \pi_{n+2}(S^2) \xrightarrow{\Sigma} \cdots$$

Some properties of these stable homotopy groups are

(1) $\pi^{stable}_{<0}(S^0) = 0$, since all groups in the sequence vanish.
(2) $\pi^{stable}_0(S^0) \cong \mathbb{Z}$, since $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.
(3) $\pi^{stable}_{>0}(S^0)$ is a finite abelian group. The finiteness was proved by Serre in his thesis.

The completion $()^\wedge_2$ that appears in the target is 2-adic completion. The only cases that matter to us are that

(1) $\mathbb{Z}^\wedge_2$ is the group of 2-adic integers, defined as the inverse limit

$$\mathbb{Z}^\wedge_2 \cong \lim(\cdots \mathbb{Z}/8 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/2).$$

This is a torsion-free group and may be thought of as a ring of power series in 2.

(2) For any finite abelian group A, $A^\wedge_2$ is just the 2-torsion subgroup.

In the Adams spectral sequence, the d_r-differential increases s (goes up) by r, and s is referred to as the Adams filtration. Because the differentials increase the total degree $t+s$ by 1, the differentials decrease t by $r-1$ and therefore decrease $t-s$ (go left) by 1. The target of the spectral sequence also now explains our display of $\mathrm{Ext}_{\mathcal{A}}$: each *column* in our "Adams chart" will tell us about a stable homotopy group.

Inspection of $E_2 \cong \mathrm{Ext}_{\mathcal{A}}$ shows that, *in the displayed range*, there are **no Adams differentials**. So we can just read off stable homotopy groups!.

Let's start with π^{stable}_0. The class $h_0 \in \mathrm{Ext}_{\mathcal{A}}$ corresponds to $2 \in \pi^{stable}_0$. Really, the E_∞-term of the spectral sequence recovers the *associated graded groups* of π^{stable}_* with respect to a filtration. In the 0-stem, this filtration is by powers of 2. When we filter $\mathbb{Z}^\wedge_2$ by powers of 2, each filtration quotient is isomorphic to $\mathbb{F}_2$, which is the series of dots in column 0.

In column 1, we just get a single $\mathbb{F}_2$, and we conclude $\pi^{stable}_1(S^0)^\wedge_2 \cong \mathbb{F}_2$. (In fact it is true before completion in the 1-stem). The element $h_1 \in \mathrm{Ext}_{\mathcal{A}}$ corresponds to $\eta \in \pi^{stable}_1$, the Hopf map.

In column 2, we just get a single $\mathbb{F}_2$, generated by h_1^2, which corresponds to $\eta^2 \in \pi^{stable}_2(S^0)$.

In column 3, the associated graded is $\mathbb{F}_2^3$. But the three dots are connected by h_0-multiplication, which tells us that in π^{stable}_3, we have nontrivial 2-extensions between these groups. We get $\pi^{stable}_3(S^0)^\wedge_2 \cong \mathbb{Z}/8$. The generator, detected by h_2, is the Hopf map ν. We learn from the Ext chart that $\eta^3 = 4\nu$. In fact, $\pi^{stable}_3(S^0)$ also has 3-torsion, and $\pi^{stable}_3(S^0) \cong \mathbb{Z}/24$.

Fri, Apr. 28

Some other generators are: $\sigma \in \pi^{stable}_7$ is a Hopf map, detected by h_3, $\varepsilon \in \pi^{stable}_8$ is detected by c_0, $\mu_9 \in \pi^{stable}_9$ is detected by Ph_1, and $\zeta_{11} \in \pi^{stable}_{11}$ is detected by Ph_2.

We include a table of the completed stable homotopy groups in this range (and the uncompleted ones, for comparison)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi^s_n(S^0)^\wedge_2$	$\mathbb{Z}^\wedge_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0
$\pi^s_n(S^0)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$

Hidden multiplications: One warning is that the multiplicative structure on the stable homotopy groups cannot always be read off of the Adams E_∞-term, again because of the associated graded issue. For example, in the Adams E_∞-term, we have $h_2^3 = h_1^2 h3$, which suggests that ν^3 should be $\eta^2 \cdot \sigma$. However, the correct relation in homotopy is

$$\nu^3 + \eta^2\sigma = \eta\varepsilon.$$

The "error" term of $\eta\varepsilon$ is detected by $h_1 c_0$, which lives in higher filtration. Thus the term in higher filtration is not seen in the associated graded.

Unfortunately, if we go out further, past the 13-stem, then the story quickly becomes more complicated. For example, there are nontrivial-differentials, such as

$$d_2(h_4) = h_0 h_3^2, \qquad d_3(h_0 h_4) = h_0 d_0, \qquad d_4(h_0^7 h_5 + d_0 e_0) = P^2 d_0, \qquad d_5(h_0^{22} h_6) P^6 d_0.$$

One of the first applications of the Adams spectral sequences came from the following result:

Theorem 4.35 (Adams)**.** *There is a division algebra (not necessarily associative) over $\mathbb{R}$ of dimension 2^n if and only if h_n is a permanent cycle in the Adams spectral sequence.*

Adams established the differentials $d_2(h_n) = h_0 h_{n-1}^2$, for $n \geq 4$.

Corollary 4.36. *The only real division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.*

There are also many more examples of "hidden multiplcations". For instance, let $\rho_{15} \in \pi_{15}$ denote a homotopy class detected by $h_0^3 h_4$. Then $h_1 \cdot h_0^3 h_4 = 0$ in the Adams spectral sequence, but $\eta \cdot \rho_{15} \neq 0$. This product is detected by Pc_0, which lives in higher filtration. As another example, let $\kappa \in \pi_{14}$ be detected by $d_0 = h_0(1)^2$ and $\overline{\kappa} \in \pi_{20}$ be detected by $g = b_{21}^2$. Then, by looking at the Adams spectral sequence, we have

$$h_2^3 \cdot d_0 = 0, \qquad h_1^2 \cdot g = 0, \qquad h_0 \cdot h_0 h_2 g = 0.$$

But in homotopy, these are all nontrivial, and we have

$$\nu^3 \cdot \kappa = \eta^3 \cdot \overline{\kappa} = 4\nu \cdot \overline{\kappa}.$$

These products are all detected by the element $h_1 P d_0$ in the Adams spectral sequence.

We also computed $\mathrm{Ext}_{\mathcal{E}(1)}$ and $\mathrm{Ext}_{\mathcal{A}(1)}$. What are these good for?

Theorem 4.37 (Adams)**.** *Let X be a finite-type CW complex. There is a (Adams) spectral sequence with*

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathrm{H}^*(X;\mathbb{F}_2), \mathbb{F}_2) \Rightarrow \pi_{t-s}^{stable}(X)_2^\wedge.$$

This even generalizes to the stable world (i.e. spectra).

Proposition 4.38. *Let ku be the (connective) complex K-theory spectrum. Then $\mathrm{H}^*(ku;\mathbb{F}_2) \cong \mathcal{A}/\!/\mathcal{E}(1)$ as an $\mathcal{A}$-module.*

Proposition 4.39. *Let ko be the (connective) real K-theory spectrum. Then $\mathrm{H}^*(ko;\mathbb{F}_2) \cong \mathcal{A}/\!/\mathcal{A}(1)$ as an $\mathcal{A}$-module.*

Proposition 4.40. *Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be a subHopf algebra. Then there is a change-of-rings isomorphism*

$$\mathrm{Ext}_{\mathcal{A}}(\mathcal{A}/\!/\mathcal{B}, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{B}}(\mathbb{F}_2, \mathbb{F}_2).$$

Combining the above results, we get spectral sequences

$$\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*^{stable}(ku)_2^\wedge, \qquad \mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*^{stable}(ko)_2^\wedge.$$

In these cases, there is no room for Adams differentials, and we read off the homotopy groups!

Proposition 4.41.

$$\pi_n(ku) \cong \begin{cases} \mathbb{Z} & n \text{ even}, \geq 0 \\ 0 & \text{else} \end{cases} \qquad \pi_n(ko) \cong \begin{cases} \mathbb{Z} & n \equiv 0,4 \pmod 8, n \geq 0 \\ \mathbb{Z}/2 & n \equiv 1,2 \pmod 8 \\ 0 & \text{else.} \end{cases}$$

The periodicity in this homotopy groups is "Bott periodicity".

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