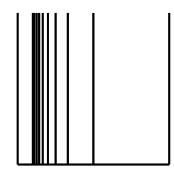
Math 651 - Topology II
Homework 1
Spring 2018

1. Recall that a subspace $A \subseteq X$ is said to be a **retract** of X if there exists a map (called a retraction) $r : X \longrightarrow A$ such that $r(a) = a$ for every $a \in A$. Show that if X is contractible and A is a retract of X, then A must also be contractible.

2. Let $A \subseteq X$. A **deformation retraction** of X onto A is a homotopy h starting at id_X such that (i) $h(x, 1) \in A$ for all x and (ii) $h(a, t) = a$ for all t. The map $h(-, 1)$ then defines a retraction $X \longrightarrow A$.

   Consider the space $X = I \times \{0\} \cup \bigcup_{x \in \{0\} \cup \{1/n\}} \{x\} \times I$ depicted to the right.

   (a) Show that X is contractible.

   (b) Show that if a space Y deformation retracts to a point $y_0 \in Y$, then for every neighborhood U of y, there is a neighborhood $V \subset U$ of y such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

   (c) Use part (b) to show that X does not deformation retract onto the point $(0, 1)$.

3. For each n, there is an equatorial inclusion $S^n \hookrightarrow S^{n+1}$. Let $S^\infty = \bigcup_n S^n$, topologized using the topology of the union. Recall that this means that $A \subseteq S^\infty$ is open or closed if and only if each $A \cap S^n$ is open or closed in S^n. Show that S^∞ is contractible.

   (Hint: Start by showing that the identity map of S^∞ is homotopic to the map

   $$(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots).$$

   Then show that the latter is nullhomotopic.)

4. Given based spaces (X, x_0) and (Y, y_0), show that

   $$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

5. Compute the fundamental group of $\mathbb{R}^2 \setminus \{0\}$.