Math 651 - Topology II
Homework II
Spring 2018

1. Show that a map $f : S^1 \longrightarrow X$ is null if and only if there exists a map $g : D^2 \longrightarrow X$ which restricts to f on $S^1 = \partial D^2$ (we say g extends the map f). (Hint: Find a homeomorphism $D^2 \cong (S^1 \times I)/(S^1 \times \{1\})$.)

2. (a) Let A be a set with two associative binary operations, $\cdot$ and $\star$. Suppose that $e \in A$ is a left and right unit for both $\cdot$ and $\star$. Finally, suppose that for any elements a, b, c, d in A, these operations satisfy

$$(a \cdot b) \star (c \cdot d) = (a \star c) \cdot (b \star d).$$

   Show that $a \cdot b = a \star b$ and that both operations are commutative.

   (b) Let G be any topological group. Use the above to show that $\pi_1(G, e)$ is necessarily abelian.

3. Does the Borsuk-Ulam theorem also hold for the torus? That is, given a map $f : S^1 \times S^1 \longrightarrow \mathbb{R}^2$, must there be a point (x, y) such that $f(x, y) = f(-x, -y)$.

4. Let X be path-connected. Show that $\pi_1(X)$ is abelian if and only if the change-of-basepoint homomorphisms $\Phi_\alpha : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ do not depend on the choice of path α.

5. A **free loop** in a space X is a map $S^1 \longrightarrow X$ with no condition on the basepoint. Since $\pi_1(X, x_0) \cong [S^1, (X, x_0)]_*$, there is a natural map

$$\Lambda : \pi_1(X, x_0) \longrightarrow [S^1, X].$$

   (a) Show that Λ is surjective if and only if X is path-connected.

   (b) Show that $\Lambda([\gamma]) = \Lambda([\delta])$ if and only if the loops γ and δ are conjugate in $\pi_1(X, x_0)$.