Math 651 - Topology II
Homework VIII
Spring 2018

1. Recall from class that if f_* and g_* are chain maps $C_* \longrightarrow D_*$, then a **chain-homotopy** between f_* and g_* is a sequence of homomorphisms $h_n : C_n \longrightarrow D_{n+1}$ such that
$$\partial^D_{n+1}(h_n(c)) + h_{n-1}(\partial^C_n c) = g(c) - f(c).$$
   (a) Given a chain complex C_*, define a new chain complex $C_* \otimes I_*$ as follows:
$$\left(C_* \otimes I_*\right)_n := C_n\{v_0\} \oplus C_n\{v_1\} \oplus C_{n-1}\{e\},$$
   and
$$\partial \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \partial(c_0) + (-1)^n c_2 \\ \partial(c_1) + (-1)^{n+1} c_2 \\ \partial(c_2) \end{pmatrix}.$$
   Check that this really defines a chain complex, meaning that $\partial^2 = 0$.

   (b) Define two chain maps $C_* \longrightarrow C_* \otimes I_*$, playing the role of ι_0 and ι_1.

   (c) Check that a chain map $h : C_* \otimes I_* \longrightarrow D_*$ corresponds to a chain homotopy between $h \circ \iota_0$ and $h \circ \iota_1$.

   (d) Show that if f_* and g_* are chain-homotopic, then f_* and g_* induce the same map on homology.

2. Find an example of a chain map that induces an isomorphism on homology but is **not** a chain-homotopy equivalence.

3. A **short exact sequence** is a sequence of homomorphisms
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$
   which is exact (has trivial homology) at each spot. A short exact sequence is called **split exact** if $B \cong A \oplus C$. Show that the following are equivalent for the (solid arrow) sequence:
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift left, "i"] & B \arrow[l, dashed, shift left, "r"] \arrow[r, shift left, "p"] & C \arrow[l, dashed, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$
   (a) The sequence is split exact

   (b) There exists a homomorphism s such that $p \circ s = \mathrm{id}_C$ (s is called a splitting)

   (c) There exists a homomorphism r such that $r \circ i = \mathrm{id}_A$ (r is called a retraction or splitting)