CLASS NOTES
MATH 651 (SPRING 2018)

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Contents

$\boxed{\textbf{Wed, Jan. 10}}$

Here are a list of main topics for this semester:

(1) the fundamental group (topology $\rightsquigarrow$ algebra) (Hatcher - Ch. 1.1; Lee - Ch. 7, Ch. 8)
(2) the theory of covering spaces (Hatcher - Ch. 1.3; Lee - Ch. 11, Ch. 12)

Example 0.1.
(a) What spaces cover $\mathbb{R}$? Only $\mathbb{R}$ itself. Every covering map $E \longrightarrow \mathbb{R}$ is a homeomorphism.
(b) What spaces cover S^1? There is the n-sheeted cover of S^1 by itself, for any nonzero integer n. (Wrap the circle around itself n times.) There is also the exponential map $\mathbb{R} \longrightarrow S^1$.
(c) What spaces cover S^2? Only S^2 itself. Every covering map $E \longrightarrow S^2$ is a homeomorphism.
(d) What spaces cover $\mathbb{RP}^2$? There is the defining quotient map $S^2 \longrightarrow \mathbb{RP}^2$ and the homeomorphisms.

(3) computation of the fundamental group via the Seifert-van Kampen theorem. (Hatcher - Ch. 1.2, Lee - Ch. 9, Ch. 10)
(4) classification of surfaces (compact, connected) and the Euler characteristic. (Lee - Ch. 6, Ch. 10)
(5) homology of CW complexes (Hatcher - Ch. 2.1, Lee - Ch. 13)

The fundamental group, an algebraic object, will turn out to be crucial for understanding topics in geometric topology (coverings, surfaces).

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Date: January 19, 2018.

1. The fundamental group - Examples

Our first major result in the course will be the computation of the fundamental group of the circle. In particular, we will show that it is nontrivial! The argument will involve a number of new ideas, and one thing I hope you will learn from this course is that **computing fundamental groups is hard**!

1.1. The fundamental group of S^1.

Today, we begin the discussion of the fundamental group of S^1. We will need the following technical result that could have been included in the fall semester.

Proposition 1.1. *(Lebesgue number lemma)[Lee, 7.18] Let $\mathcal{U}$ be an open cover of a compact metric space X. Then there is a number $\delta > 0$ such that any subset $A \subseteq X$ of diameter less than δ is contained in an open set from the cover.*

For any n, consider the loop in S^1 given by $\gamma_n(t) = e^{2\pi int}$. For today, we will denote the standard basepoint of S^1, the point $(1, 0)$, by the symbol $\star$.

Theorem 1.2. *The assignment $n \mapsto \gamma_n$ is an isomorphism of groups*

$$\Gamma : \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, \star).$$

Proof. Let's start by showing that it is a homomorphism. First note that γ_0 is the constant path at 1, which is the identity element of the fundamental group. Also, note that γ_{-n} is the path-inverse of γ_n. It then remains to show that the path $\gamma_n \cdot \gamma_k$ is path-homotopic to γ_{n+k} when n and k are non-negative.

For any $0 \leq c \leq 1$, we can define a path which first traverses γ_n on the time interval $[0, c]$ and then traverses γ_k on the time interval $[c, 1]$. Any two choices of c gives homotopic paths. The choice $c = 1/2$ gives the usual path-composition $\gamma_n \cdot \gamma_k$, whereas the choice $c = n/(n + k)$ gives γ_{n+k}.

To show that Γ is also a bijection, we will rely on the exponential map

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ t &\mapsto e^{2\pi it}. \end{aligned}$$

Note that $p^{-1}(\star) = \mathbb{Z}$. One important property of this map that we will need is that we can cover S^1, say using the open sets $U_1 = S^1 \setminus \{(1, 0)\}$ and $U_2 = S^1 \setminus \{(-1, 0)\}$. On each of these open sets U_i, the preimage $p^{-1}(U_i)$ is a (countably infinite) disjoint union of subsets $V_{i,j}$ of $\mathbb{R}$, and p restricts to a homeomorphism $p : V_{i,j} \cong U_i$.

If $f : X \longrightarrow S^1$ is a map from some space X, then by a **lift** $\tilde{f} : X \longrightarrow \mathbb{R}$ we mean simply a map such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Fri, Jan. 12

Lemma 1.3. *Let $\gamma : I \longrightarrow S^1$ be a loop at $\star$ and let $n \in \mathbb{Z}$. Then there is a unique lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ such that $\tilde{\gamma}(0) = n$.*

Proof. By the Lebesgue number lemma applied to I, there is a subdivision of I into subintervals $[s_i, s_{i+1}]$ such that each subinterval is contained in a single $\gamma^{-1}(U_i)$.

Consider the first such subinterval $[0, s_1] \subseteq \gamma^{-1}(U_2)$. Now our lifting problem simplifies to that on the right. The interval $[0, s_1]$ is connected, so the image of $\tilde{\gamma}$ must lie in a single component $V_{1,j}$. And we have no choice of the component since we have already decided that $\tilde{\gamma}(0)$ must be n. Call the component $V_{2,0}$.

$$\begin{tikzcd} & \amalg V_2, j \arrow[d, "p"] \\ {[0, s_1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & U_2 \end{tikzcd}$$

Now our lifting problem reduces to lifting against the *homeomorphism* $p_{2,0} : V_{2,0} \cong U_2$, and we define our lift on $[0, s_1]$ to be the composite $p_{2,0}^{-1} \circ \gamma$. Now play the same game with the next interval $[s_1, s_2]$. We already have a lift at the point s_1, so this forces the choice of component at this stage. By induction, at each stage we have a unique choice of lift on the subinterval $[s_k, s_{k+1}]$. Piecing these all together gives the desired lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$. ■

Thus given a loop γ at $\star$, there is a unique lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ that starts at 0. The endpoint of the lift $\tilde{\gamma}$ must also be in $p^{-1}(0) = \mathbb{Z}$. We claim that the function $\gamma \mapsto w(\gamma) = \tilde{\gamma}(1)$ is inverse to Γ. First we must show it is well-defined.

Lemma 1.4. *Let $h : \gamma \simeq_p \delta$ be a path-homotopy between loops at $\star$ in S^1. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow \mathbb{R}$ such that $\tilde{h}(0,0) = 0$.*

Proof. We already know about the unique lift $\tilde{\gamma}$ on $I \times 0$. On $0 \times I$, the only possible lift is the constant lift. Now use the Lebesgue number lemma again to subdivide the compact square $I \times I$ so that every subsquare is mapped by γ into one of the U_i. Using the same argument as above, we get a unique lift on each subsquare, starting from the bottom left square and moving along each row systematically. ■

Note that the lift $\tilde{h}$ is a path-homotopy between the lifts $\tilde{\gamma}$ and $\tilde{\delta}$. This is because $\tilde{h}(0,t)$ and $\tilde{h}(1,t)$ are lifts of constant paths. By the uniqueness of lifts, according to Lemma 1.3, the lift of a constant path must be a constant path. It follows that $\tilde{\gamma}(1) = \tilde{\delta}(1)$. This shows that the function $w : \pi_1(S^1) \longrightarrow \mathbb{Z}$ is well-defined.

It remains to show that w is the inverse of Γ.

First note that $\delta_n(s) = ns$ is a path in $\mathbb{R}$ starting at 0, and $p \circ \delta_n(s) = e^{2\pi i(ns)} = \gamma_n(s)$, so δ_n is a lift of γ_n starting at 0. By uniqueness of lifts (Lemma 1.3), δ_n must be $\tilde{\gamma_n}$. Therefore

$$w \circ \Gamma(n) = w(\gamma_n) = \tilde{\gamma}_n(1) = \delta(1) = n.$$

It remains to check that $\Big[\Gamma(w(\gamma))\Big] = [\gamma]$ for any loop γ. Consider lifts $\Gamma\widetilde{(w(\gamma))}$ and $\tilde{\gamma}$. These are both paths in $\mathbb{R}$ starting at 0 and ending at $\tilde{\gamma}(1) = w(\gamma)$ (this uses that $w \circ \Gamma(n) = n$). But any two such paths are homotopic (use a straight-line homotopy)! Composing that homotopy with the exponential map p will produce a path-homotopy $\Gamma(w(\gamma)) \simeq_p \gamma$ as desired. ■