$\boxed{\textbf{Mon, Mar. 19}}$

Example 3.41. (Surface of genus g) Similarly, if we take a connect sum of g tori, we get the surface of genus g, M_g. It has fundamental group

$$\pi_1(M_g) \cong F(a_1, b_1, \ldots, a_g, b_g)/[a_1, b_1] \ldots [a_g, b_g].$$

We now have $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$.

We are headed towards a "classification theorem" for compact surfaces, so let us now show that if $g_1 \neq g_2$ then M_{g_1} is not homeomorphic to M_{g_2}. We show this by showing they have different fundamental groups. As we have said already, understanding a group given by a list of generators and relations is not always easy, so we make life easier by considering the **abelianizations** of the fundamental groups.

The abelianization G_{ab} of G is the group defined by

$$G_{ab} = G/[G, G],$$

where $[G, G]$ is the (normal) subgroup generated by commutators.

Lemma 3.42. *The abelianization $F(a_1, \ldots, a_n)_{ab}$ is the free abelian group $\mathbb{Z}^n$.*

Proof. We already did this in the case $n = 2$ for understanding the fundamental group of the torus, and the proof generalizes. ■

The abelianization is characterized by a universal property. For a group G, let $q : G \longrightarrow G_{ab}$ be the quotient map. Then the universal property of the quotient gives the following result.

Proposition 3.43. *Let G be a group and A an abelian group. Then any homomorphism $\varphi : G \longrightarrow A$ factors uniquely as $G \xrightarrow{q} G_{ab} \xrightarrow{\overline{\varphi}} A$.*

When we apply this to the surface M_g, we get

Proposition 3.44. $\pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$.

Proof. Let $F = F(a_1, b_1, \ldots, a_n, b_n)$, let $N \trianglelefteq F$ be the normal subgroup generated by (i.e. the normal closure of) the product $[a_1, b_1] \ldots [a_g, b_g]$, and let $G = \pi_1(M_g) \cong F/N$. Since the quotient map $q : F \longrightarrow G$ is surjective, it follows that $q([F, F]) = [G, G]$. By the Third Isomorphism Theorem, we get

$$G_{ab} := G/[G, G] = G/q([F, F]) \cong F/[F, F] \cong \mathbb{Z}^{2g}.$$

■

Lemma 3.45. *If $H \cong G$ then $H_{ab} \cong G_{ab}$.*

As a result, we see that if $g_1 \neq g_2$ then $\pi_1(M_{g_1}) \neq \pi_1(M_{g_2})$ because their abelianizations are not isomorphic.

Corollary 3.46. *If $g_1 \neq g_2$ them $M_{g_1} \ncong M_{g_2}$.*

Note that we have also distinguished all of these from S^2 (which has trivial fundamental group) and from $\mathbb{RP}^2$ (which has abelian fundamental group $\mathbb{Z}/2\mathbb{Z}$).

What about the Klein bottle K? We found before the break that $\pi_1(K) \cong F(a, b)/aba^{-1}b$. If we abelianize this fundamental group, we get

Proposition 3.47. *The abelianized fundamental group of the Klein bottle is*

$$\pi_1(K)_{ab} \cong (\mathbb{Z}\{a\} \times \mathbb{Z}\{b\})/(a + b - a + b) = \mathbb{Z}\{a\} \times \mathbb{Z}\{b\}/2b \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Proof. The idea is the same as in the previous examples. Under the quotient $F(a,b) \longrightarrow \mathbb{Z}\{a\} \times \mathbb{Z}\{b\}$, the element $aba^{-1}b$ is sent to $a+b-a+b$ (this is simply changing from multiplicative notation to additive notation. ∎

This group is different from all of the others, so K is not homeomorphic to any of the above surfaces. The last main example is

Example 3.48. ($\mathbb{RP}^2\#\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2$) Suppose we take a connect sum of g copies of $\mathbb{RP}^2$. We will call this surface N_g. Following the previous examples, we see that we get a fundamental group of

$$\pi_1(N_g) \cong F(a_1,\ldots,a_g)/a_1^2a_2^2\ldots a_g^2$$

and $\chi(N_g) = 1-g+1 = 2-g$. The abelianization is then

$$\pi_1(N_g)_{ab} \cong \mathbb{Z}^g/(2,2,\ldots,2).$$

Define a homomorphism $\varphi : \mathbb{Z}^g/(2,\ldots,2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{g-1}$ by

$$\varphi(n_1,\ldots,n_g) = (n_1, n_2-n_1, n_3-n_1,\ldots,n_g-n_1).$$

Then it is easily verified that φ is an isomorphism. In other words,

$$\pi_1(N_g)_{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}^{g-1}.$$

Ok, so we have argued that the compact surfaces S^2, M_g ($g \geq 1$), and N_g ($g \geq 1$) all have different fundamental groups and thus are not homeomorphic. The remarkable fact is that these are *all* of the compact (connected) surfaces.

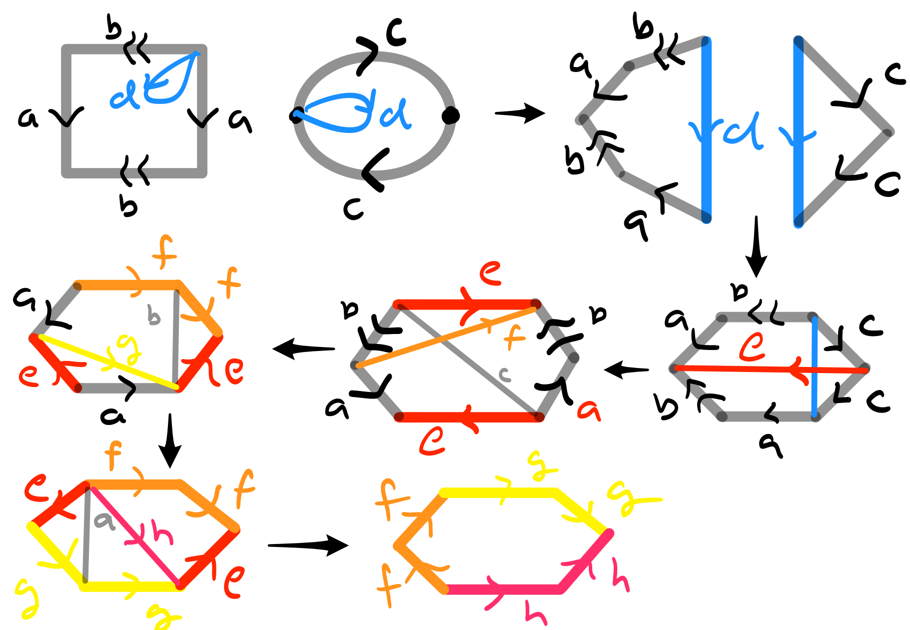
Theorem 3.49. *Every compact, connected surface is homeomorphic to some M_g, $g \geq 0$ or to some N_g, $g \geq 1$.*

Corollary 3.50. *If $\chi(M) = n$ is odd, then $M \cong N_{2-n}$*

All of these examples are formed by taking connected sums of T^2's or $\mathbb{RP}^2$'s. What happens if we mix them?

Lemma 3.51. *$T^2\#\mathbb{RP}^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$.*

In other words, one bad apple spoils the whole bunch. The proof is in the picture:



In particular, this implies that $M_g \# N_k \cong N_{2g+k}$.

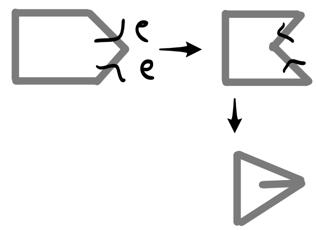
Proof of the theorem. Let M be a compact, connected surface. **We assume without proof** (see Prop 6.14 from Lee) that

- M is a 2-cell complex with a single 2-cell.
- the attaching map $\alpha : S^1 \longrightarrow M^1$ for the 2-cell has the following property: let U be the interior of a 1-cell. Then the restriction $\alpha : \alpha^{-1}(U) \longrightarrow U$ is a double cover. In other words, if we label ∂D^2 according to the edge identifications as we have done in the examples, each edge appears exactly twice. Note that this must happen since each interior point on the edge needs to have a half-disk on two sides.

So we can visualize M as a quotient of a $2n$-sided polygon.

As we said above, each edge appears exactly twice on the boundary of the two-cell. If the two occurrences have **opposite** orientations (as in the sphere), we say the pair is an **oriented pair**. If the two occurrences have the **same** orientation (as in $\mathbb{RP}^2$), we say this is a **twisted** pair. There will be 4 reductions in the proof!!

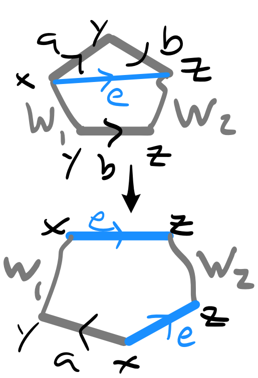
(1) If $M \cong S^2$, we are done, so suppose (for the rest of the proof) this is *not* the case. Then we can reduce to a cell structure with no *adjacent oriented pairs*. (Just fold these together.)

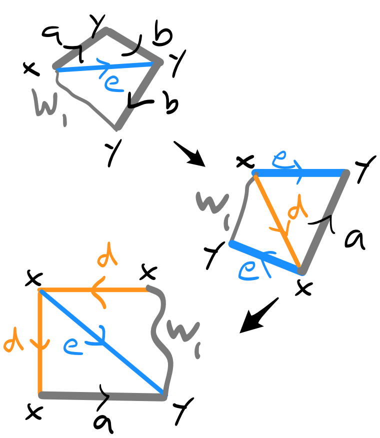


(2) We can reduce to a cell structure where all twisted pairs are adjacent.

If this creates any adjacent oriented pairs, fold them in.

(3) We can reduce to a cell structure with a single 0-cell. Suppose a is an edge from x to y and that $x \neq y$. Let b be the other edge connecting to y. By (1), b can't be a^{-1}. If $b = a$ then $x = y$. Suppose $b \neq a$, and write z for the other vertex on b. Then the edge b must occur somewhere else on the boundary. We use the moves in the pictures below, depending on whether the pair b is oriented or twisted.





This converts a vertex y into a vertex x. Note that this procedure does not separate any adjacent twisted pairs, since the adjacent twisted pair b gets replaced by d.

(4) Observe that any oriented pair a, a^{-1} is interlaced with another oriented pair b, b^{-1}. If not, we can write the boundary in the form $aW_1a^{-1}W_2$. Now, given our assumption and previous steps, no edge in W_1 gets identified with an edge in W_2. It follows that if the endpoints of a are x and y, then these two vertices never get identified with each other, as the vertex x cannot appear in W_1 and similarly y cannot appear in W_2.

(5) We can further arrange it so that there is no interference: the oriented pairs of edges occur as $aba^{-1}b^{-1}$ with no other edges in between. The proof is in the picture below, taken from p. 177 of Lee.

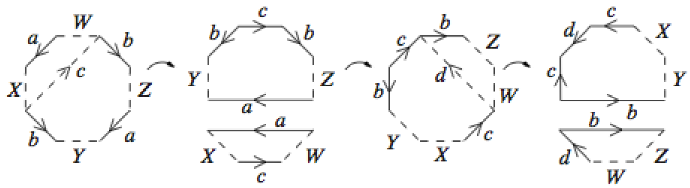


Fig. 6.22: Bringing intertwined complementary pairs together.

Now we are done by Lemma 3.51. M is homeomorphic either to a connect sum of projective planes or to a connect sum of tori. ■

Fri, Mar. 23

We saw in Corollary 3.50 that if $\chi(M)$ is odd, we can immediately identify the homeomorphism type of M. If $\chi(M)$ is even, this is not the case, as T^2 and K both have Euler characteristic equal to 0. To handle the even case, we make a definition.

Say that a surface M is **orientable** if it has a cell structure as above with no twisted pairs of edges.

Proposition 3.52. *A surface is orientable if and only if it is homeomorphic to some M_g.*

Proof. ($\Leftarrow$) Our standard cell structures for these surfaces have no twisted pairs of edges. ($\Rightarrow$) Apply the algorithm described in the above proof, starting with only oriented pairs of edges. Step 1 does not introduce any new edges. Step 2 can be skipped. Steps 3 cuts-and-pastes along a pair of oriented edges and so does not change the orientation of any edges. Step 4 does not change the surface. Step 5 again only cuts-and-pastes along oriented edges. It follows that in reducing to standard form, we do not introduce any twisted pairs of edges. ■

In fact, you should be able to convince yourself that a surface is orientable if and only if *every* cell structure as above has no twisted pairs. The point is that if you start with a cell structure involving some twisted pairs and you perform the reductions described in the proof, you will never get rid of any twisted pairs of edges.

The fact that the M_g can be embedded in $\mathbb{R}^3$ whereas the N_g cannot is precisely related to orientability. In general, you can embed a (smooth) n-dimensional manifold in $\mathbb{R}^{2n}$, but you can improve this to $\mathbb{R}^{2n-1}$ if the manifold is orientable. The definition we have given here depends on particular kinds of CW structures, but other definitions of orientability (in terms of homology) apply more widely.

In addition to the N_g's, the Möbius band is a 2-manifold that is famously non-orientable.

4. Higher homotopy groups

We have just been studying surfaces and have determined (well, at least given presentations for) their fundamental groups. We have also seen (on exam 1) that there are higher homotopy groups $\pi_n(X)$, so we might ask about the groups $\pi_n(M_g)$ and $\pi_n(N_k)$.

Recall, again from the exam, that any covering $E \longrightarrow B$ induces an isomorphism on all higher homotopy groups. So it suffices to understand the universal covers of these surfaces.

The first example would be $M_0 = S^2$, which is simply-connected. Note that this space is also the universal cover of $N_1 = \mathbb{RP}^2$, so these will have the same higher homotopy groups. We will come back to these on Monday.

Another example is the componentwise-exponential covering $q \times q : \mathbb{R}^2 \longrightarrow T^2$, which shows that T^2 has no higher homotopy groups. Note that we also could have deduced this using that

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$$

and that S^1 has no higher homotopy groups (also from Exam 1).

What about the Klein bottle K? Well, consider the relation on T^2 given by $(x, y) \sim (x + \frac{1}{2}, 1 - y)$. The quotient $T^2/\sim$ is K, and the quotient map $T^2 \longrightarrow K$ is a double cover. It follows that the universal cover of T^2, which is $\mathbb{R}^2$, is also the universal cover of K. So K also has no higher homotopy groups!

For the surfaces of higher genus, we start by generalizing the double cover $T^2 \longrightarrow K$.

Proposition 4.1. *If $g \geq 1$, then there is a double cover of N_g by M_{g-1}.*

Lemma 4.2. *Suppose that $p : E \longrightarrow B$ is a double cover of a surface B, and let W be another surface. Then there is a double cover $E\#W\#W \longrightarrow B\#W$.*

The lemma implies the proposition as follows:

Proof. We already know about the double cover $S^2 \longrightarrow \mathbb{RP}^2$, which is the case $g = 1$. Recall (Lemma 3.51) that $N_3 \cong \mathbb{RP}^2\#T^2$. By the lemma, we get a double cover $M_2 \cong S^2\#T^2\#T^2 \longrightarrow \mathbb{RP}^2\#T^2 \cong N_3$. By tacking on more copies of T^2, this handles the case of g odd.

We also discussed above the double cover $T^2 \longrightarrow K$, which is the case $g = 2$. By the lemma, we get a double cover $M_3 \cong T^2\#T^2\#T^2 \longrightarrow K\#T^2 \cong N_4$. By tacking on more copies of T^2, this handles the case of g even. ■