Mon, Mar. 26

Last time, we saw that there is a double covering of the nonorientable surface N_g by the orientable surface M_{g-1}. It remains to find the universal cover of M_{g-1}.

Proposition 4.3. *For $g \geq 1$, the universal cover of M_g is $\mathbb{R}^2$.*

Sketch. We have already shown this for $g = 1$. In the higher genus case, this is more difficult. This is sometimes described using “hyperbolic” geometry. In that approach, $\mathbb{R}^2$ is replaced by the (homeomorphic) upper half-place, equipped with the hyperbolic metric. The idea is that you can tile the hyperbolic half-plane by polygons. Since M_g has a presentation as an (oriented) quotient of a polygon, this establishes a covering of M_g by the half-plane. ∎

Ok, so we know that $\pi_n(\mathbb{RP}^2) \cong \pi_n(S^2)$. What are these groups? We will show later that $\pi_2(S^2) \cong \mathbb{Z}$. Just like for S^1, a generator for this group is the identity map $S^2 \longrightarrow S^2$. But the fascinating thing is that, in contrast to S^1, there are plenty of interesting higher homotopy groups! Here is a table of homotopy groups of spheres, taken from Wikipedia.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

There are several things to note in this table.

(1) We have $\pi_n(S^3) = \pi_n(S^2)$ for $n \geq 3$. There is a map $S^3 \longrightarrow S^2$ that induces this isomorphism on homotopy groups. It is the Hopf map η we studied before $(\mathbb{C}^2 - \{0\} \longrightarrow \mathbb{CP}^1)$. This map is not a cover, since the fibers are circles. But this is a higher analogue of a covering: it is an S^1-bundle. The analogue of the “evenly covered neighborhoos” here is called “local triviality” of the bundle. This means that each point in $x \in \mathbb{CP}^1$ has a neighborhood U such that $\eta^{-1}(U) \cong S^1 \times U$. Remembering that a point in $\mathbb{CP}^1$ is of the form $x = [z_1 : z_2]$, consider the open sets $U_1 = \{[z_1 : z_2] | z_1 \neq 0\}$ and $U_2 = \{[z_1 : z_2] | z_2 \neq 0\}$. These certainly cover $\mathbb{CP}^1$, and the isomorphism

$$\eta^{-1}(U_1) \cong S^1 \times U_1$$

is

$$(z_1, z_2) \mapsto \left(\frac{z_1}{\|z_1\|}, [z_1 : z_2] \right).$$

A bundle still has a lifting property for paths and homotopies, but the lifts are no longer unique. This means that we can't necessarily lift an arbitrary map $Y \longrightarrow S^2$ up to a map $Y \longrightarrow S^3$, and it need not be true that *all* higher homotopy groups of S^2 are identified with those of S^3. It turns out that what happens here is that we have a "long exact sequence" relating the homotopy groups of S^3, S^2, and S^1 (most of which are trivial).

(2) We have $\pi_n(S^k) = 0$ if $n < k$. The argument is similar to the one that showed the higher spheres are all simply-connected. The main step is to show that any map $S^n \longrightarrow S^k$ is homotopic to a nonsurjective map if $n < k$.

(3) The answers are eventually constant on each diagonal. There is a suspension homomorphism $\pi_n(S^k) \longrightarrow \pi_{n+1}(S^{k+1})$ that induces these isomorphisms. The stable answer for $\pi_{k+n}(S^k)$ is known as the nth stable homotopy group of spheres and is written π_n^s. We have

$$\pi_0^s = \mathbb{Z}, \qquad \pi_1^s = \mathbb{Z}/2, \qquad \pi_2^s = \mathbb{Z}/2, \qquad \pi_3^s = \mathbb{Z}/24.$$

These groups are known out to around $n = 60$.

(4) Most of the unstable groups are finite. The only infinite ones are $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{4k-1}(S^{2k})$. The latter are all $\mathbb{Z}\times$(finite group). This is a theorem of J. P. Serre. This implies that all of the stable groups are finite, except $\pi_0^s = \mathbb{Z}$.

Ok, so homotopy groups are hard! But there are a few more examples of spaces whose homotopy groups are all known, so let's mention those before we abandon all hope and despair.

Example 4.4. Remember that we have a double cover $S^n \longrightarrow \mathbb{RP}^n$ inducing an isomorphism on all higher homotopy groups. But S^n does not have any homotopy groups until π_n, so this means that $\pi_k(\mathbb{RP}^n) = 0$ if $1 < k < n$. The inclusion $S^n \hookrightarrow S^{n+1}$, $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0)$ induces an inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$. As n gets higher, we lose more and more homotopy groups. In the limit, $S^\infty = \bigcup_n S^n$ has no homotopy groups (and in fact it is contractible). Similarly, $\mathbb{RP}^\infty$ has only a fundamental group of $\mathbb{Z}/2$ but no higher homotopy groups.

Example 4.5. There is an analogous story for $\mathbb{CP}^n$. Here, we have for every n, an S^1-bundle $S^{2n-1} \simeq \mathbb{C}^n - \{0\} \longrightarrow \mathbb{CP}^n$. This map induces an isomorphism on π_k for $k \geq 3$ and gives $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$. So the only nontrivial homotopy group of $\mathbb{CP}^\infty$ is $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$.

Wed, Mar. 28

Last time, we discussed higher homotopy groups of some familiar spaces. We saw that most of the M_g and N_g have no higher homotopy groups. On the other hand, basic spaces like S^2 and $\mathbb{RP}^2$ have very complicated (and unknown) higher homotopy groups. The other examples in which we had complete understanding of the higher homotopy groups were the infinite-dimensional complexes $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$. It turns out that this is quite typical: a finite cell complex almost always has infinitely many nontrivial homotopy groups!

5. Homology

This is rather disheartening. We think of a cell complex as an essentially finite amount of information. It would be nice if we only got finitely many algebraic objects out of it. There is such a construction: homology. As we will see, this will combine a number of the ideas we have recently encountered: the fundamental group and Euler characteristics. A good way to think about homology is that it is a more sophisticated version of the Euler characteristic.

We will deal with two versions of homology. The first, **singular homology**, is a good theoretical tool that is convenient for proving theorems. But it is not great for doing actual calculations. For that purpose, we will also consider **cellular homology**, which is defined for CW complexes. **Simplicial homology** is yet another version which is convenient for calculation, though we will not consider this version in our course.

5.1. **Singular homology.** Let Δ^n denote the standard n-simplex, which can be defined as

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, \quad t_i \geq 0\}.$$

We will denote by $v_i \in \Delta^n$ the vertex defined by $t_i = 1$ and $t_j = 0$ if $j \neq i$. Note that each "facet" of the simplex, in which we have restricted one of the coordinates to zero, is an $(n-1)$-dimensional simplex. More generally, if we set k of the coordinates equal to zero, we get a face which is an $(n-k)$-dimensional simplex.

Definition 5.1. Let X be a space. A **singular n-simplex** of X will simply be a continuous map $\Delta^n \longrightarrow X$.

Let $C_n^{\mathrm{Sing}}(X)$, or simply $C_n(X)$, be the free abelian group on the set of singular n-simplices of X. An element of $C_n(X)$ is referred to as a (singular) n-chain on X. Our goal is to assemble the $C_n(X)$, as n varies, into a "chain complex"

$$\dots \longrightarrow C_3(X) \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X).$$

To say that this is a chain complex just means that composing two successive maps in the sequence gives 0.

Fri, Mar. 30

Exam day