$\boxed{\textbf{Mon, Apr. 2}}$

We wish to specify a homomorphism

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X).$$

Since $C_n(X)$ is a free abelian group, the homomorphism ∂_n is completely specified by its value on each generator, namely each n-simplex.

There are $n+1$ standard inclusions $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$, given by inserting 0 in position i in Δ^n.

Definition 5.2. The **singular boundary homomorphism**

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

is defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [\sigma \circ d^i].$$

Example 5.3.

(1) If σ is a 1-simplex (from v_0 to v_1), then

$$\partial_1(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] = [v_1] - [v_0].$$

(2) If σ is a 2-simplex with vertices v_0, v_1, and v_2, and edges e_{01}, e_{02}, and e_{12}, then

$$\partial_2(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] + [\sigma \circ d^2] = [e_{12}] - [e_{02}] + [e_{01}]$$

The claim is that this defines a chain complex. The signs have been inserted into the definition to make this work out.

Proposition 5.4. *The boundary squares to zero, in the sense that $\partial_{n-1} \circ \partial_n = 0$.*

Proof. We will use

Lemma 5.5. *For $i > j$, the composite*

$$\Delta^{n-2} \xrightarrow{d^j} \Delta^{n-1} \xrightarrow{d^i} \Delta^n \quad \textit{is equal to the composite} \quad \Delta^{n-2} \xrightarrow{d^{i-1}} \Delta^{n-1} \xrightarrow{d^j} \Delta^n.$$

Consider the case $i = 3$, $j = 1$, $n = 4$. We have

$$d^3(d^1(t_1, t_2, t_3)) = d^3(t_1, 0, t_2, t_3) = (t_1, 0, t_2, 0, t_3) = d^1(t_1, t_2, 0, t_3) = d^1(d^2(t_1, t_2, t_3)).$$

This argument generalizes.

For the proposition,

$$\begin{aligned}
\partial_{n-1}\Big(\partial_n(\sigma)\Big) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i[\sigma\circ d^i]\right)\\
&= \sum_{i=0}^{n}(-1)^i\,\partial_{n-1}([\sigma\circ d^i])\\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j[\sigma\circ d^i\circ d^j]\\
&= \sum_{i=0}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]+\sum_{i=0}^{n}\sum_{j\geq i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
\text{(changing bounds)}\quad &= \sum_{i=1}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]+\sum_{i=0}^{n-1}\sum_{j\geq i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
\text{(Lemma)}\quad &= \sum_{i=1}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma\circ d^j\circ d^{i-1}]+\sum_{i=0}^{n-1}\sum_{j\geq i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
&= -\sum_{j=0}^{n-1}\sum_{i-1\geq j}(-1)^j(-1)^{i-1}[\sigma\circ d^i\circ d^j]+\sum_{i=0}^{n-1}\sum_{j\geq i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
&= 0.
\end{aligned}$$

■

We have shown that any two successive simplicial boundary homomorphisms compose to zero, so that we have a chain complex. What do we do with a chain complex? Take homology!

Definition 5.6. If

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots$$

is a chain complex, then we define the **nth homology group** $\mathrm{H}_n(C_*, \partial_*)$ to be

$$\mathrm{H}_n(C_*, \partial_*) := \ker\partial_n / \operatorname{im}\partial_{n+1}.$$

Note that the fact that $\partial_n \circ \partial_{n+1} = 0$ implies that $\operatorname{im}\partial_{n+1}$ is a subgroup of $\ker\partial_n$, so that the definition makes sense. A complex (C_*, ∂_*) is said to be **exact** at C_n if we have equality $\ker\partial_n = \operatorname{im}\partial_{n+1}$. Thus the homology group $\mathrm{H}_n(C_*, \partial_*)$ "measures the failure of C_* to be exact at C_n."

Definition 5.7. Given a space X, we define the **singular homology groups** of X to be

$$\mathrm{H}_n(X;\mathbb{Z}) := \mathrm{H}_n(C_*(X), \partial_*).$$

Note that we only defined the groups $C_n(X)$ for $n \geq 0$. For some purposes, it is convenient to allow chain groups C_n for negative values of n, so we declare that $C_n(X) = 0$ for $n < 0$. This means that $\ker\partial_0 = C_0(X)$, so that $\mathrm{H}_0 = C_0(X)/\operatorname{im}\partial_1 = \operatorname{coker}(\partial_1)$.

Terminology: The group $\ker\partial_n$ is also known as the group of **n-cycles** and sometimes written Z_n. The group $\operatorname{im}(\partial_{n+1})$ is also known as the group of **boundaries** and sometimes written B_n.

$\boxed{\textbf{Wed, Apr. 4}}$

Remark 5.8. It is worth noting that since each $C_n(X)$ is free abelian and $\ker\partial_n$ and $\operatorname{im}\partial_{n+1}$ are both subgroups, they are necessarily also free abelian.

Example 5.9. Consider $X = *$. Then $C_n(\{*\}) = \mathbb{Z}\{\mathbf{Top}(\Delta^n, \{*\})\} \cong \mathbb{Z}$ for all n. The differential $\partial_n : C_n(\{*\}) \longrightarrow C_{n-1}(\{*\})$ takes the (constant) singular n-simplex c_n to the alternating sum

$$\sum_i (-1)^i c_{n-1} = \begin{cases} c_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

In other words, the chain complex is

$$\dots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

so that the only nonzero homology group is $\mathrm{H}_0(*) \cong \mathbb{Z}$.

But already for $X = \Delta^1$, the chain groups are infinite rank, and computing becomes impractical. On the other hand, the singular homology groups have good properties. For starters, we will discuss functoriality.

Given a map $f : X \longrightarrow Y$, we can compose any singular n-simplex of X with f to get a singular n-simplex of Y. This produces a function

$$f_n : \mathrm{Sing}_n(X) \longrightarrow \mathrm{Sing}_n(Y)$$

and therefore a homomorphism

$$f_n : C_n(X) \longrightarrow C_n(Y).$$

It remains to discuss how this interacts with homology.

Definition 5.10. Let (C_*, ∂_*^C) and (D_*, ∂_*^D) be chain complexes. Then a **chain map** $f_* : (C_*, \partial_*^C) \longrightarrow (D_*, \partial_*^D)$ is a sequence of homomorphisms $f_n : C_n \longrightarrow D_n$, for each n, such that each diagram

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial_n^C"'] & D_n \arrow[d, "\partial_n^D"] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \end{tikzcd}$$

commutes for each n.

Since f_n is given by post-composition with f, whereas each term of ∂_n is given by precomposing with the face inclusions, it follows that the homomorphisms (f_*) on the singular chains assemble to produce a chain map.

We set up this definition in order to get

Proposition 5.11. *A chain map $f_* : (C_*, \partial_*^C) \longrightarrow (D_*, \partial_*^D)$ induces homomorphisms $f_n : \mathrm{H}_n(C_*, \partial_*^C) \longrightarrow \mathrm{H}_n(D_*, \partial_*^D)$ for each n.*

Proof. Let $x \in C_n$ be a cycle, meaning that $\partial^C(x) = 0$. Then $\partial^D(f_n(x)) = f_{n-1}(\partial^C(x)) = f_{n-1}(0) = 0$, so that $f_n(x)$ is a cycle in D_n. In order to get a well-defined map on homology, we need to show that if x is in the image of ∂_{n+1}^C, then $f_n(x)$ is in the image of ∂_{n+1}^D. But if $x = \partial_{n+1}^C(y)$, then $f_n(x) = f_n(\partial_{n+1}^C(y)) = \partial_{n+1}^D f_{n+1}(y)$, which shows that $f_n(x)$ is a boundary. ∎

There is an obvious way to compose chain maps, so that chain complexes and chain maps form a category $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$.

Proposition 5.12. *The assignment $X \mapsto (C_*(X), \partial_*)$ and $f \mapsto f_*$ defines a functor*

$$C_* : \mathbf{Top} \longrightarrow \mathbf{Ch}_{\geq 0}(\mathbb{Z}).$$

Given the above discussion, it only remains to show that this construction takes identity morphisms to identity morphisms and that it preserves composition. We leave this as an exercise.

Note that the sequence of homology groups $\mathrm{H}_n(C_*, \partial_*^C)$ of a chain complex is not quite a chain complex, since there are no differentials between the homology groups. You can think of this as a degenerate case of a chain complex, in which all differentials are zero. But it is more common to simply call this a **graded abelian group**. If X_* and Y_* are graded abelian groups, then a graded map $f_* : X_* \longrightarrow Y_*$ is simply a collection of homomorphisms $f_n : X_n \longrightarrow Y_n$. Graded maps compose in the obvious way, so that we get a category **GrAb** of graded abelian groups. Then Proposition 5.11 is the main step in proving

Proposition 5.13. *Homology defines a functor*

$$\mathrm{H}_* : \mathbf{Ch}_{\geq \mathbf{0}}(\mathbb{Z}) \longrightarrow \mathbf{GrAb}.$$

The composition of two functors is always a functor. Thus Proposition 5.12 and Proposition 5.13 combine to yield

Proposition 5.14. *Singular homology defines a functor*

$$\mathrm{H}_*^{\mathrm{Sing}} : \mathbf{Top} \longrightarrow \mathbf{GrAb}.$$

This implies, for instance, that homeomorphic spaces have isomorphic singular homology groups.

Fri, Apr. 6

Last time, we discussed how a map of spaces induces a map on homology. Previously, we saw that the induced map on fundamental groups only depended on the homotopy class of the map, and we might ask the same question here.

Proposition 5.15. *Suppose that $f \simeq g$ as maps $X \longrightarrow Y$. Then f and g induce the same map on homology.*

Corollary 5.16. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then f induces an isomorphism on homology.*

Sketch of Proposition 5.15. See Theorem 13.8 of Lee for complete details.

If we have maps $f, g : X \longrightarrow Y$, it would be enough to show that their difference $f_* - g_*$ at the level of chains always takes values in the group of boundaries. Unfortunately, this is not always true, but it turns out to be true on cycles, which is enough to deduce the proposition. For simplicity, we consider the "universal" case, in which $Y = X \times I$ and f and g are the inclusions at time 0 and 1, respectively.

The idea is to define a homomorphism (called a "chain-homotopy") $h_n : C_n(X) \longrightarrow C_{n+1}(X \times I)$ for all n, satisfying the equation

$$h \circ \partial + \partial \circ h = g_* - f_*.$$

If you plug in a cycle x to this formula, you learn that $g_*(x) - f_*(x)$ is a boundary, so that f_* and g_* agree at the level of homology.

When $n = 0$, we simply take $h_0(x)$ to be the constand path in $X \times I$ from $(x, 0)$ to $(x, 1)$. At level 1, if σ is a path in X, we wish to define $h_1(\sigma) \in C_2(X \times I)$ with

$$h_0(\sigma(1) - \sigma(0)) + \partial \circ h_1(\sigma) = \sigma \times \{1\} - \sigma \times \{0\}.$$

We take $h_1(\sigma)$ to be the formal difference of simplices with vertices $(\sigma_0, 0)$, $(\sigma_1, 0)$, and $(\sigma_1, 1)$ and $(\sigma_0, 0)$, $(\sigma_0, 1)$, $(\sigma_1, 1)$. Similar formulas work in higher dimensions. ∎

Example 5.17. We saw that the one-point space has homology groups nonvanishing only in dimension zero, given by the group $\mathbb{Z}$. It follows that the same is true for any contractible space, such as I^n or D^n or $\mathbb{R}^n$.