Mon, Apr. 9

5.2. **The functor** $\mathrm{H}_0(-)$.

Proposition 5.18. *If X is path-connected and nonempty, then $\mathrm{H}_0(X) \cong \mathbb{Z}$.*

Proof. Define $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ by sending each 0-simplex (i.e. point of X) to 1. As X is nonempty, the map ε is surjective. We claim that $\ker(\varepsilon) = B_0 = \mathrm{im}(\partial_1)$.

For any 1-simplex σ, $\partial_1(\sigma) = \sigma(1) - \sigma(0)$, so $\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$. This shows that $B_0 \subseteq \ker(\varepsilon)$.

Now suppose that $c = \sum_{i=1}^k n_i x_i$ is a 0-chain. Pick a point $x_0 \in X$, and, for each $i = 1, \dots, k$, pick a path $\alpha_i : x_0 \rightsquigarrow x_i$. Then $\partial_1(\alpha_i) = x_i - x_0$, so that $x_i \equiv x_0$ in $C_0(X)/B_0(X)$. Therefore $c \equiv (\sum_i n_i)x_0$ in $C_0(X)/B_0$. Now if $c \in \ker(\varepsilon)$, this means that $\sum_i n_i = 0$, so that $c \equiv 0$ in $C_0(X)/B_0$. In other words, $c \in B_0$. ■

To describe H_0 for a general space, we first discuss how path components interact with homology.

Proposition 5.19. *Let $\{X_\alpha\}$ be the set of path-components of X and $\iota_\alpha : X_\alpha \longrightarrow X$ the inclusions. These induce an isomorphism*

$$\bigoplus_\alpha \mathrm{H}_*(X_\alpha) \cong \mathrm{H}_*(X).$$

Proof. Since the image of any singular n-simplex must be contained in a single path-component, we get already a splitting of the chain complexes

$$\bigoplus_\alpha C_*(X_\alpha) \cong C_*(X).$$

This produces the splitting on the level of homology. ■

Corollary 5.20. *For any space X, $\mathrm{H}_0(X)$ is free abelian on the set of path-components of X. In other words,*

$$\mathrm{H}_0(X) \cong \mathbb{Z}\{\pi_0(X)\}.$$

5.3. **The Mayer-Vietoris Sequence.**

One of the fundamental tools for computing homology is the Mayer-Vietoris sequence, which is analogous to the van Kampen theorem for the fundamental group. First, some terminology.

Recall (from just before Definition 5.7) that we say that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if it has no homology, meaning that $\mathrm{im}(f) = \ker(g)$. Very often, we encounter an exact sequence in which either A or C is 0. If $A = 0$, then the image of f must also be zero, so that g must be injective. Similarly, if $C = 0$, then the kernel of g must be all of B, so that f must be surjective. For a longer sequence, such as $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow \dots$, we say it is exact if it is so at each group in the sequence.

We consider a space X with open subsets U and V. We will denote the inclusions as in the diagram

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\ & V \arrow[ur, "\ell"'] & \end{tikzcd}$$

Theorem 5.21 (Mayer-Vietoris long exact sequence)**.** *Let X be a space, and let U and V be open subsets with $U \cup V = X$. Then there is a long exact sequence in homology*

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(U \cap V) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(U) \oplus \mathrm{H}_n(V) \xrightarrow{k_* - \ell_*} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(U \cap V) \xrightarrow{i_* \oplus j_*} \dots$$

Before proving the theorem, we give a sample application.

Example 5.22. ($\mathrm{H}_*(S^k)$) Combining Example 5.9 with Proposition 5.19 gives that

$$\mathrm{H}_i(S^0) \cong \begin{cases} \mathbb{Z}^2 & i = 0 \\ 0 & \text{else.} \end{cases}.$$

Wed, Apr. 11

We use the Mayer-Vietoris sequence to compute the homology of the higher spheres. We argue by induction that for $k > 0$,

$$\mathrm{H}_i(S^k) \cong \begin{cases} \mathbb{Z} & i = 0, k \\ 0 & \text{else.} \end{cases}.$$

The base case is S^1. Take U and V to be the open subsets of S^1 given by removing the north and south poles, respectively. Notice that U and V are both contractible and that $U \cap V$ deformation retracts to the equatorial S^0. Thus the Mayer-Vietoris sequence becomes

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(S^0) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(*) \oplus \mathrm{H}_n(*) \xrightarrow{k_* - \ell_*} \mathrm{H}_n(S^1) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(S^0) \xrightarrow{i_* \oplus j_*} \dots.$$

Note that when n is larger than 1, then $\mathrm{H}^n(S^1)$ is flanked by two zero groups and must therefore by zero. We are left then only with the exact sequence

$$0 \longrightarrow \mathrm{H}_1(S^1) \xrightarrow{\partial_1} \mathrm{H}_0(S^0) \xrightarrow{i_* \oplus j_*} \mathrm{H}_0(*) \oplus \mathrm{H}_0(*) \xrightarrow{k_* - \ell_*} \mathrm{H}_0(S^1) \cong \mathbb{Z} \longrightarrow 0.$$

This becomes

$$0 \longrightarrow \mathrm{H}_1(S^1) \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{(1\ -1)} \mathbb{Z} \longrightarrow 0.$$

It follows that the image of ∂_1 is the subgroup generated by $(1, -1)$, so that $\mathrm{H}_1(S^1) \cong \mathbb{Z}$.

Now for the induction step, suppose the formula holds for $\mathrm{H}_*(S^k)$ and consider S^{k+1}. We again take U and V to be the complements of the poles in S^{k+1}. Now the Mayer-Vietoris sequence becomes

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(S^k) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(*) \oplus \mathrm{H}_n(*) \xrightarrow{k_* - \ell_*} \mathrm{H}_n(S^{k+1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(S^k) \xrightarrow{i_* \oplus j_*} \dots.$$

We know by Proposition 5.18 that $\mathrm{H}_0(S^{k+1}) \cong \mathbb{Z}$, and the exact sequence gives that $\mathrm{H}_{n+1}(S^{k+1}) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(S^k)$ is an isomorphism for $n \geq 1$. Finally, the group $\mathrm{H}_1(S^{k+1})$ is in the exact sequence

$$0 \longrightarrow \mathrm{H}_1(S^{k+1}) \xrightarrow{\partial_1} \mathrm{H}_0(S^k) \xrightarrow{i_* \oplus j_*} \mathrm{H}_0(*) \oplus \mathrm{H}_0(*) \longrightarrow \mathrm{H}_0.$$

The map $i_* \oplus j_*$ is the diagonal map $\mathbb{Z} \longrightarrow \mathbb{Z}^2$, which is injective. It follows that $\mathrm{H}_1(S^{k+1}) = 0$.

Fri, Apr. 13

The main step in the proof of the Mayer-Vietoris theorem is the following result. We say that a sequence $0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{q} C_* \longrightarrow 0$ of chain complexes is exact if each sequence $0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{q} C_n \longrightarrow 0$ is exact.

Proposition 5.23. *A short exact sequence* $0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{q} C_* \longrightarrow 0$ *of chain complexes induces a long exact sequence in homology*

$$\ldots \longrightarrow \mathrm{H}_{n+1}(C) \xrightarrow{\delta} \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{q_*} \mathrm{H}_n(C) \xrightarrow{\delta} \mathrm{H}_{n-1}(A) \longrightarrow \ldots$$

Proof. We start with the construction of the "connecting homomorphism δ". Thus let $c \in C_n$ be a cycle. Choose a lift $b \in B_n$, meaning that $q(b) = c$. We then have $q(\partial_n(b)) = \partial_n(q(b)) = \partial_n(c) = 0$. Since the rows are exact, we have $\partial_n(b) = i(a)$ for some unique $a \in A_{n-1}$, and we define

$$\delta(c) := a.$$

$$\begin{tikzcd} & b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ a \arrow[r, mapsto] & \partial(b) \arrow[r, mapsto] & 0 \end{tikzcd}$$

It remains to see how a depends on the choice of b. Thus let $d \in \ker(q)$, so that $q(b+d) = c$. By exactness, we have $d = i(e)$ for some $e \in A_n$. Then

$$i(a + \partial_n(e)) = \partial_n(b) + i(\partial_n(e)) = \partial_n(b) + \partial_n(i(e)) = \partial_n(b) + \partial_n(d) = \partial_n(b+d),$$

so that $\delta(c) = a + \partial_n(e) \sim a$. In other words, a specifies a well-defined homology class.

Since we want δ to be well-defined not only on cycles but also on homology, we need to show that if c is a boundary, then $\delta(c) \sim 0$. Thus suppose $c = \partial(c')$. We can then choose b' such that $q(b') = c'$. It follows that $\partial(b')$ would be a suitable choice for b. But then $\partial(b) = \partial(\partial(b')) = 0$, so that $\delta(c) = 0$.

Exactness at B: First, we see that $q_* \circ i_* = 0$ since this is already true at the chain level. Now suppose that $b \in \ker(q_*)$. This means that $q(b) = \partial(c)$ for some $c \in C_{n+1}$. Now choose a lift $d \in B_{n+1}$ of c. Then we know

$$q(\partial(d)) = \partial(q(d)) = \partial(c) = q(b).$$

In other words, $q(b - \partial(d)) = 0$, so that we must have $b - \partial(d) = i(a)$ for some a. Since $b \sim b - \partial(d)$, we are done.

Exactness at C: We first show that $\delta \circ q_* = 0$. Thus let $b \in B_n$ be a cycle. We wish to show that $\delta(q_*(b)) = 0$. But the first step in constructing $\delta(q(b))$ is to choose a lift for $q(b)$, which we can of course take to be b. Then $\partial(b) = 0$, so that $a = 0$ as well.

Now suppose that $c \in C_n$ is a cycle that lives in the kernel of δ. This means that $a = \partial(e)$ for some e. But then $b - i(e)$ is a cycle, and $q(b - i(e)) = c$, so c is in the image of q_*.

Exactness at A: First, we show that $i_* \circ \delta = 0$. Let $c \in C_n$ be a cycle. Then if $\delta(c) = a$, then by construction, we have $i(a) = \partial(b) \sim 0$, so that $i_* \circ \delta = 0$.

Finally, suppose that $a \in A_n$ is a cycle that lives in $\ker i_*$. Then $i(a) = \partial(b)$ for some b, but then $a = \delta(q(b))$. ∎

Sketch of Theorem 5.21. We would like to apply Proposition 5.23 to the sequence

$$0 \longrightarrow C_*(U \cap V) \xrightarrow{i_* + j_*} C_*(U) \oplus C_*(V) \xrightarrow{k_* - \ell_*} C_*(X) \longrightarrow 0.$$

The problem is that this is not exact at $C_*(X)$. The reason is that not every singular n-simplex in X is contained entirely in U or V. Instead, we introduce the subcomplex $C_*^{U,V}(X)$, where $C_n^{U,V}(X)$ is the free abelian group on simplices which are entirely contained in either U or V.

We claim that the inclusion $C_*^{U,V}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence. We need to define a homotopy inverse $f : C_*(X) \longrightarrow C_*^{U,V}(X)$. The idea is to use "barycentric subdivision". The subdivision of an n-simplex expresses it as the union of smaller n-simplices. By the Lebesgue Number Lemma, repeated barycentric subdivision will eventually decompose any singular n-simplex of X into a collection of n-simplices, each of which is either contained in A or in B. This subdivision allows you to define a chain map f. You then show that subdivision of simplices is chain-homotopic to the identity. See Proposition 2.21 of Hatcher for a much more detailed discussion. ■