**Mon, Apr. 16**

5.4. **The Hurewicz Theorem.** We saw previously that $\mathrm{H}_0(X) \cong \mathbb{Z}\{\pi_0(X)\}$. What about $\mathrm{H}_1(X)$? It turns out this is closely related to $\pi_1(X)$. First note that given a map $\alpha : S^1 \longrightarrow X$, we get an induced map $\mathbb{Z} \cong \mathrm{H}_1(S^1) \longrightarrow \mathrm{H}_1(X)$. If we pick a preferred generator for $\mathrm{H}_1(S^1)$, for example the 1-simplex $\Delta^1 \longrightarrow S^1$ which is the quotient map

$$\Delta^1 \cong I \longrightarrow I/\partial I \cong S^1,$$

then this picks out a particular element of $\mathrm{H}_1(X)$.

**Proposition 5.24.** *This element $\alpha_*(1) \in \mathrm{H}_1(X)$ only depends on the homotopy class of $\alpha$.*

*Proof.* This follows from Proposition 5.15. ■

We then define the **Hurewicz** function

$$h : \pi_1(X, x_0) \longrightarrow \mathrm{H}_1(X)$$

by $h([\alpha]) = \alpha_*(1)$. By the proposition, this is well-defined on homotopy-classes.

**Theorem 5.25** (Hurewicz)**.** *Assume that $X$ is path-connected. Then $h$ induces an isomorphism*

$$H_1(X) \cong \pi_1(X, x_0)_{ab}.$$

*Proof.* We first show that $h$ is a group homomorphism. First, it preserves identity elements since if we consider the constant loop at $x_0$ as a 1-cycle, we can express it as the boundary of the constant 2-simplex at $x_0$. Next, suppose we have two loops $\alpha$ and $\beta$. We wish to show that $h(\alpha \cdot \beta) = h(\alpha) + h(\beta)$. Either by using the Square Lemma (Lemma 3.16) or by writing one down explicitly, we can define a 2-simplex $\sigma_{\alpha,\beta}$ whose restriction to the boundary is the three edges $\alpha$, $\alpha\beta$, and $\beta$. Then $\partial(\sigma_{\alpha,\beta}) = \alpha - \alpha \cdot \beta + \beta$. This shows that $h(\alpha \cdot \beta) = h(\alpha) + h(\beta)$.

Since we now know that $h$ is a homomorphism, we can use the universal property of abelianization to factor

$$h : \pi_1(X) \longrightarrow \mathrm{H}_1(X)$$

through $\hat{h} : \pi_1(X)_{ab} \longrightarrow \mathrm{H}_1(X)$. It remains to show that $\hat{h}$ is bijective.

(Surjectivity): For each $x \in X$, pick a path $p_x : x_0 \rightsquigarrow x$. We also write $p : C_0(X) \longrightarrow C_1(X)$ for the resulting function. Now for each 1-simplex $a$, we can define a loop $\tilde{a}$ at $x_0$ by $p_{a(0)} \cdot a \cdot \overline{p_{a(1)}}$. Then

$$h(\tilde{a}) = [p_{a(0)}] + [a] + [\overline{p_{a(1)}}] = [a] + [p \circ \partial(a)].$$

Now take an arbitrary 1-cycle $c = \sum_i n_i a_i$. Then we get

$$h(\tilde{a_1}^{n_1} \tilde{a_2}^{n_2} \cdots \tilde{a_k}^{n_k}) = \sum_i n_i[a_i] + n_i[p \circ \partial(a_i)] = [c] + p([\partial(c)]) = [c]$$

since $c$ was assumed to be a cycle.

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(Injectivity): Let $\alpha \in \pi_1(X)$ be in the kernel of $h$. We wish to show that $\alpha$ is trivial in $\pi_1(X)_{ab}$. If $h(\alpha) = 0$, this means that $\alpha$, when considered as a 1-simplex, is a boundary. Suppose, for example, that

$$\alpha = \partial(\sigma)$$

for some 2-simplex $\sigma : \Delta^2 \longrightarrow X$. But $\partial(\sigma) = \sigma_{1,2} - \sigma_{0,2} + \sigma_{0,1}$, so if this is equal to $\alpha$ in $C_1(X)$, then $\alpha$ must be either $\sigma_{0,1}$ or $\sigma_{1,2}$, and the other of these edges must agree with $\sigma_{0,2}$. Write $\beta$ for the path $\sigma_{0,1}$. Then, by the square lemma, the two-simplex $\sigma$ gives rise to a path-homotopy $\alpha\beta \simeq_p \beta$. In other words, $\alpha \simeq_p c_{x_0}$.

The trouble is that, in general, there is no reason to expect $\alpha$ to be the differential on a single 2-simplex. Rather, we expect to have

$$\alpha = \partial(\sum n_i \sigma_i).$$

Again, from the square lemma, each of these 2-simplices $\sigma_i$ will give rise to a path-homotopy. All of the faces of the $\sigma_i$'s cancel in $C_1(X)$, to leave only $\alpha$. If we try to do the same manipulation in $\pi_1(X)$, using the path-homotopies, we need to allow ourselves to commute elements, since this can happen in $C_1(X)$ to allow for the cancellation there. So if we abelianize $\pi_1(X)$, we can perform the same cancellation to show that $[\alpha] = [c_{x_0}] \in \pi_1(X)_{ab}$. ■

**Example 5.26.** Recall from Proposition 3.44 that $\pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$. It follows that

$$\mathrm{H}_1(M_g) \cong \mathbb{Z}^{2g}.$$

**Example 5.27.** Recall from Example 3.48 that $\pi_1(N_g)_{ab} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows that

$$\mathrm{H}_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In fact, a stronger version of the Hurewicz theorem holds. We will not prove the stronger version.

**Theorem 5.28.** *Suppose that $\pi_k(X) = 0$ for $k < n$, where $n > 1$. Then $\mathrm{H}_n(X) \cong \pi_n(X)$.*

**Corollary 5.29.** *Let $n > 1$. Then* $\pi_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & 0 < k < n. \end{cases}$.

*Proof.* We already showed that this is the homology of the sphere. Since $S^n$ is simply connected, Theorem 5.28 gives that $\pi_2(S^n) \cong \mathrm{H}_2(S^n)$. If $n = 2$, this is $\mathbb{Z}$ and we are done. If $n > 2$, this is 0, and then we apply the Hurewicz theorem at level 3. Repeat until you reach the first nonzero homology group. ■

5.5. **Cellular homology.** While singular homology is defined for all spaces and is nicely functorial, it is not so practical for computing by hand. For this purpose, we introduce cellular homology, which is defined for CW complexes.

Recall that at the end of last semester, we defined the **degree** of a map $f : S^1 \longrightarrow S^1$ by considering the induced map on fundamental groups. This map is multiplication by some integer, which we called the degree. If $f$ was not based, the definition of degree involved the change-of-basepoint homomorphism. But now that we know about (singular) homology, there is a simpler definition, which works equally well in higher dimensions.

**Definition 5.30.** Let $f : S^n \longrightarrow S^n$ be any map. for $n \geq 1$. Then the induced map on homology

$$f_* : \mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n)$$

is multiplication by some integer $d$. We define the **degree** of $f$ to be this integer $d$.

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**Definition 5.31.** Let $X$ be a CW complex. Define the group $C_n^{cell}(X)$ of cellular $n$-chains by

$$C_n^{cell} := \mathbb{Z}\{n\text{-cells of } X\}.$$

To specify the differential $d_n : C_n(X) \longrightarrow C_{n-1}(X)$, we need to give the coefficients in

$$d_n(f) = \sum n_i e_i.$$

Here $f$ is an $n$-cell, which is described by its attaching map $S^{n-1} \xrightarrow{f} \mathrm{sk}_{n-1}X$. The coefficient $n_i$ in the expansion is the degree of the map

$$S^{n-1} \xrightarrow{f} X^{n-1} \twoheadrightarrow X^{n-1}/X^{n-2} \cong \bigvee S^{n-1} \xrightarrow{e_i} S^{n-1}.$$

This works well if $n-1 \geq 1$. The $d_1$ is defined similarly. A 1-cell $e$ is determined by the attaching map, which simply specifies the endpoints $e(1)$ and $e(0)$. We define $d_1(e) = e(1) - e(0)$.

We now define the **cellular homology groups** to be the homology of this complex:

$$\mathrm{H}_n^{cell}(X) := \mathrm{H}_n(C_*^{cell}(X)).$$

On the face of it, this definition does not make sense, since we have not verified that $d \circ d = 0$. Probably the simplest way to establish this is to recognize that $C_n^{cell}(X) \cong \mathrm{H}_n(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X))$. Then the cellular differential can be viewed as the connecting homomorphism in a Mayer-Vietoris sequence. See the dicsussion above Theorem 2.35 of Hatcher for more details.

This definition of homology might sound complicated, but in practice it is quite simple. For instance, if our CW complex has a single 0-cell, then each 1-cell must be a loop, and the $d_1$-differential is just zero. Another immediate consequence of the definition is the following.

**Proposition 5.32.** *Suppose that $X$ is an $n$-dimensional CW complex. Then $\mathrm{H}_k^{cell}(X) = 0$ for $k > n$.*

This is simply because $X$ has no cells above dimension $n$, so that $C_k^{cell}(X) = 0$ if $k > n$.

Let's look at some examples.

**Example 5.33.** Take $X = S^2$. Pick the CW structure having a single vertex and a single 2-cell. Then $C_1(X) = 0$, so both $d_2$ and $d_1$ must be the zero map. The chain complex $C_*(S^2)$ is

$$\mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z}.$$

Here we get $H_0 = H_2 = \mathbb{Z}$ and $H_1 = 0$. The same would for any $S^n$, with $n \geq 2$.

**Example 5.34.** Take $X = S^2$. Pick the CW structure having a single vertex, a single edge, and two 2-cells attached via the identity map $S^1 \cong S^1$. Then $C_0(S^2) = C_1(S^2) = \mathbb{Z}$ and $C_2(S^2) = \mathbb{Z}^2$. The map

$$d_1 : C_1 = \mathbb{Z} \longrightarrow C_0 = \mathbb{Z}$$

is $d_1(e) = 0$ since the edge $e$ is a loop. If we write $f_1$ and $f_2$ for the 2-cells, we see that $d_2(f_1) = d_2(f_2) = e$. Thus the resulting chain complex is

$$\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Here we see that $H_0 \cong \mathbb{Z}$ since $d_1 = 0$, so that $B_0 = 0$ and $H_0 = Z_0 = \mathbb{Z}$. Next, the statement $d_1 = 0$ also means that $Z_1 = C_1 = \mathbb{Z}$, and we see that $d_2$ is surjective, so that $B_1 = Z_1 = C_1$. It follows that $H_1 \cong \mathbb{Z}$. Finally, the kernel of $d_2$ is the cyclic subgroup of $\mathbb{Z}^2$ generated by $(1, -1)$, so $H_2 = Z_2 \cong \mathbb{Z}$.

**Example 5.35.** Take $X = S^2$. Pick the CW structure having two cells in each degree $\leq 2$. Here each attaching map $S^{n-1} \longrightarrow X^{n-1}$ is an identity map. Write $x_1$ and $x_2$ for the vertices and $e_1$ and $e_2$ for the edges. We have $d_1(e_i) = x_2 - x_1$. Similarly, we have $d_2(f_i) = e_1 - e_2$. The resulting chain complex is

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2$$

Here, the differential $d_1$ has image the subgroup generated by $(-1, 1)$, so $H_0 \cong \mathbb{Z}^2/(-1, 1) \cong \mathbb{Z}$. The kernel of $d_1$ is the subgroup generated by $(1, -1)$, which is the image of $d_2$, so $H_1 = 0$. The kernel of $d_2$ is again the subgroup generated by $(-1, 1)$, so that $H_2 \cong \mathbb{Z}$.