Mon, Apr. 23

In the examples on Friday, we saw that it did not matter which CW structure on S^2 we chose. In each case, we got the same answer, and these answers also agreed with the singular homology groups.

Theorem 5.36. *Let X be a space equipped with a choice of CW structure. Then*

$$\mathrm{H}_n^{cell}(X) \cong \mathrm{H}_n^{Sing}(X)$$

for all n.

Since the right-hand side does not depend on any choice of CW structure, the left-hand side must not either.

We do not give the proof (see Hatcher, Theorem 2.35). The idea is to first recognize that $\mathrm{H}_n^{Sing}(X) \cong \mathrm{H}_n^{Sing}(\mathrm{sk}_{n+1}X)$. Then we have

$$\mathrm{H}_n^{Sing}(X) \cong \mathrm{H}_n^{Sing}(\mathrm{sk}_{n+1}X) \twoheadleftarrow \mathrm{H}_n^{Sing}(\mathrm{sk}_n X) \longrightarrow \mathrm{H}_n^{Sing}(\mathrm{sk}_n X)/\mathrm{sk}_{n-1}X) \cong C_n^{cell}(X).$$

You show that this map lands in the subgroup $Z_n^{cell}(X)$ and induces an isomorphism to the quotient $Z_n^{cell}(X)/B_n^{cell}(X)$.

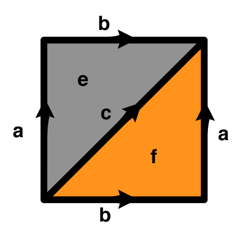
Example 5.37. Take $X = T^2$. The standard cell structure we have used has a single 0, two 1-cells a and b, and a single 2-cell e attached via $aba^{-1}b^{-1}$. Since there is a single 0-cell, this means that automatically $d_1 = 0$. To calculate $d_2(e)$, we wish to calculate the coefficient in front of a and b. For a, we must compose the attaching map $aba^{-1}b^{-1}$ with the projection onto the circle a. This means all of the b's are sent to 0, so in the end we have $aa^{-1} = 0$. The same goes for b, so $d_2 = 0$. The chain complex $C_*(T^2)$ is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}.$$

Since all differentials are zero in $C_*(T^2)$, it is immediate that

$$H_0(T^2) \cong \mathbb{Z}, \qquad H_1(T^2) \cong \mathbb{Z}^2, \qquad H_2(T^2) \cong \mathbb{Z}.$$

Example 5.38. (torus, second approach) Consider the CW structure on T^2 as given in the picture to the right. The resulting chain complex is



$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

We read off right away that $H_0(T^2) \cong \mathbb{Z}$. Then

$$H_1(T^2) = Z^3/\operatorname{im}(d_2) = \mathbb{Z}^3/\mathbb{Z}(1,1,-1) \cong \mathbb{Z}^2.$$

For the last isomorphism, note that since $(1,1,-1) \in \mathbb{Z}^3$ is linearly independent from $(0,1,0)$ and $(0,0,1)$, we can take these three elements as generators of the group $\mathbb{Z}^3$. It follows that the quotient is $\mathbb{Z}^2$. Finally,

$$H_2(T^2) = \ker(d_2) = \mathbb{Z}(1,1) \cong \mathbb{Z}.$$

There are a few algebraic results that are quite helpful in doing these computations.

Theorem 5.39. *(Fundamental theorem for finitely generated abelian groups) If A is a finitely generated abelian group, then*

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$$

for some non-negative integers r and k and positive integers $n_1, \ldots, n_k$.

Theorem 5.40. *(Smith normal form) Let A be an $n \times k$ matrix with integer values. Then, by using column and row operations, A can be reduced to*

$$A \sim \begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $n_i \mid n_{i+1}$. This is the **Smith normal form** *for the matrix.*

If a differential d_n is represented by A, then you reduce A to normal form, and the kernel of d_n will be (isomorphic to) $\mathbb{Z}^j$, where j is the number of zero columns in the normal form.

Wed, Apr. 25

Example 5.41. ($\mathbb{RP}^2$) We have a CW structure with a single cell in dimensions 0, 1, and 2. The attaching map for the 2-cell is $\gamma_2 : S^1 \longrightarrow S^1$. It follows that the chain complex $C_*(\mathbb{RP}^2)$ is

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Thus $H_0(\mathbb{RP}^2) \cong \mathbb{Z}$, $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, and $H_2(\mathbb{RP}^2) = 0$.

Example 5.42. (Klein bottle, first version) Recall that we have a CW structure on K having a single 0-cell and 2-cell and two 1-cells. The 2-cell is attached according to the relation $aba^{-1}b$. It follows that $C_*(K)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\binom{0}{2}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}$$

We read off immediately that $H_0(K) \cong \mathbb{Z}$ and that $H_2(K) = 0$ since d_2 is injective. The remaining calculation is

$$H_1(K) = \mathbb{Z}^2/\mathbb{Z}(0,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Example 5.43. (Klein bottle, second version) Recall that we discussed a second CW structure on K having a single 0-cell and 2-cell and two 1-cells. The 2-cell is attached according to the relation c^2d^2. It follows that $C_*(K)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\binom{2}{2}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}$$

We read off immediately that $H_0(K) \cong \mathbb{Z}$ and that $H_2(K) = 0$ since d_2 is injective. The remaining calculation is

$$H_1(K) = \mathbb{Z}^2/\mathbb{Z}(2,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Here the isomorphism $\mathbb{Z}^2/\mathbb{Z}(2,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is induced by the map

$$\mathbb{Z}^2 \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$
$$(n,k) \mapsto (n-k,k).$$

Example 5.44. (Orientable surfaces) We have a CW structure on M_g with a single 0-cell and 2-cell and $2g$ 1-cells. The attaching map for the 2-cell is the product of commutators $[a_1, b_1] \dots [a_g, b_g]$. It follows that $C_*(M_g)$ is the chain complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}.$$

So $H_0(M_g) \cong Z$, $H_1(M_g) \cong \mathbb{Z}^{2g}$, and $H_2(M_g) \cong \mathbb{Z}$.

Example 5.45. (Nonorientable surfaces) We have a CW structure on N_g with a single 0-cell and 2-cell and g 1-cells. The attaching map for the 2-cell is the product $a_1^2 \dots a_g^2$. It follows that $C_*(N_g)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}} \mathbb{Z}^g \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

So $H_0(N_g) \cong \mathbb{Z}$, $H_1(N_g) \cong \mathbb{Z}^g/\mathbb{Z}(2, \dots, 2) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$, and $H_2(N_g) = 0$. Again, the isomorphism $\mathbb{Z}^g/\mathbb{Z}(2, \dots, 2) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$ is induced by

$$\mathbb{Z}^g \twoheadrightarrow \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$$
$$(n_1, \dots, n_g) \mapsto (n_1 - n_g, n_2 - n_g, \dots, n_{g-1} - n_g, n_g).$$

Remark 5.46. According to the previous examples and our Proposition 3.52, a compact, connected surface M satisfies $H_2(M) \cong \mathbb{Z}$ if M is orientable and satisfies $H_2(M) = 0$ if M is not orientable.

So, for a surface, H_2 tells us about orientability.

We have seen that cellular homology tends to be quite computable, so what is the drawback? One major drawback is functoriality. Recall that any map of spaces $f : X \longrightarrow Y$ gave us a map on singular homology. For cellular homology, this is only true if the map is compatible with the CW structures, in the sense that f carries the n-skeleton of X into the n-skeleton of Y **for all** n. Such maps are called **cellular**, and this is a very strong condition. In fact, any map is homotopic to a cellular map, but in general finding a cellular approximation to a given map is quite nontrivial.

Example 5.47. Now let's consider $\mathbb{RP}^n$ for $n > 2$. The cellular chain complex is

$$\begin{array}{ccccccccc} C_n & \xrightarrow{1+(-1)^n} & C_{n-1} & \longrightarrow & \dots & \xrightarrow{2} & C_1 & \xrightarrow{0} & C_0 \\ \| & & \| & & & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z} & & & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

To understand the differential d_k, it suffices to understand what it does to the k-cell e_k. The attaching map for this k-cell is the double cover $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$. Then $d_k(e_k) = n_k e_{k-1}$, where n_k is the degree of the map

$$S^{k-1} \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}.$$

To visualize this, think of $\mathbb{RP}^{k-1}$ as the quotient of the northern hemisphere of S^{k-1} by a relation on the boundary. Then $\mathbb{RP}^{k-2}$ is the quotient of the boundary, so the quotient $\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ is the northern hemisphere with the equator collapsed. The map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ factors through $S^{k-1}/S^{k-2} \cong S^{k-1} \vee S^{k-1}$. The map on the nothern hemisphere $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$ is the identity. On the other hand, the map on the southern hemisphere can be identified with the map $(x_1, \dots, x_k) \mapsto (-x_1, \dots, -x_k)$. This is a homeomorphism, so the question is whether it is homotopic to the identity, in which case the map on this hemisphere corresponds to 1, or it is not, in which case the maps corresponds to -1. But this map is a sequence of k reflections, each of which has determinant -1. So the map has determinant $(-1)^k$. This number then agrees with the degree of the map, and we find that $n_k = 1 + (-1)^k$.

It follows that in degrees less than n we have

$$H_{2i}(\mathbb{RP}^n) = 0, i > 0, \qquad H_0(\mathbb{RP}^n) = \mathbb{Z}, \qquad H_{2i+1}(\mathbb{RP}^n) = \mathbb{Z}/2.$$

To determine $H_n(\mathbb{RP}^n)$, we consider $d_n : C_n \longrightarrow C_{n-1}$. If n is even, then d_n is injective, so $H_n(\mathbb{RP}^n) = 0$. On the other hand, if n is odd, then $d_n = 0$, so that $H_n(\mathbb{RP}^n) \cong \mathbb{Z}$.

The Euler characteristic computation according to homology is now

$$\chi(\mathbb{RP}^{2k}) = 0 + 0 + \cdots + 0 + 1 = 1, \qquad \chi(\mathbb{RP}^{2k+1}) = 1 + 0 + 0 + \cdots + 0 + 1 = 2.$$

Recall that we mentioned that for an n-manifold, the top homology group $H_n(M)$ is either $\mathbb{Z}$ or 0, depending on whether the manifold is orientable or not. The above shows that $\mathbb{RP}^n$ is orientable if and only if n is odd ($n \geq 1$).

Fri, Apr 27

Example 5.48. We can also consider $X = \mathbb{CP}^n$. But this turns out to be much easier, since $\mathbb{CP}^n$ only has cells in even degrees. There can't possibly be any nonzero differentials! We then read off that

$$\mathrm{H}_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n \text{ \& } k \text{ even} \\ 0 & \text{else.} \end{cases}$$

We also have $\chi(\mathbb{CP}^n) = n + 1$, and $\mathbb{CP}^n$ is always orientable.

Recall that we talked about the Euler characteristic for surfaces. For any chain complex C_*, we define the Euler characteristic of C_* by $\chi(C_*) = \sum(-1)^i \operatorname{rank}(C_i)$ (when this sum makes sense). Recall that the **rank** of a free abelian group is the maximal number of linearly independent elements. For example, if $C \cong \mathbb{Z}^r \oplus A$, where A is finite, then $\operatorname{rank} C = r$.

Lemma 5.49. *Suppose given a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of finitely-generated abelian groups. Then

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Proof. We show that $\operatorname{rank}(B) \geq \operatorname{rank}(A) + \operatorname{rank}(C)$ and leave the other direction as an exercise. Let $a_1, \ldots, a_r$ be a maximal linearly independent set in A and $c_1, \ldots, c_s$ a maximal linearly independent set in C. Since g is surjective, we can lift these elements to $\tilde{c}_i \in B$. We claim that the collection $\{f(a_i)\} \cup \{\tilde{c}_j\}$ is linearly independent. Thus consider an equation

$$\sum_i m_i f(a_i) + \sum_k n_k \tilde{c}_k = 0.$$

By applying g, we get

$$\sum_k n_k c_k = 0.$$

Since the c_k's are independent, we conclude that $n_k = 0$ for all k. Since f is injective, we now learn that

$$\sum_i m_i a_i = 0.$$

But since the a_i's are independent, we learn that $m_i = 0$ for all i. This shows that $\{f(a_i)\} \cup \{\tilde{c}_j\}$ is independent. ∎

Proposition 5.50. *For any chain complex, we have $\chi(C_*) = \chi(H_*(C_*))$.*

Proof. The key is to note that we have short exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0.$$

and

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0.$$

By a Lemma 5.49, these tell us that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$

and

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i).$$

So

$$\sum_i (-1)^i \operatorname{rank}(C_i) = \sum_i (-1)^i (\operatorname{rank}(B_i) + \operatorname{rank}(H_i) + \operatorname{rank}(B_{i-1})).$$

This is a telescoping sum, and we end up with $\chi(H_*)$. ■

As an example, we talked about the homology of $\mathbb{RP}^2$ earlier. We saw this was

$$H_0(\mathbb{RP}^2) \cong \mathbb{Z}, \qquad H_1(\mathbb{RP}^2) = \mathbb{Z}/2, \qquad H_2(\mathbb{RP}^2) = 0.$$

Since the standard model for $\mathbb{RP}^2$ has no cells above dimension 2, there is of course no homology in higher dimensions. The Euler characteristic computation according to homology is

$$\chi(\mathbb{RP}^2) = \operatorname{rank}(\mathbb{Z}) - \operatorname{rank}(\mathbb{Z}/2) = 1.$$

Proposition 5.50 tells us that the Euler characteristic only depends on the homology of the space, not on the particular cellular model.