Wed, Jan. 17

Using problem 4 from Homework I, we get the following result.

Corollary 1.5. *Let T^n denote the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times). Then $\pi_1(T^n) \cong \mathbb{Z}^n$.*

Theorem 1.6. *(Borsuk-Ulam Theorem) For every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, there is an antipodal pair of points $\{x, -x\} \subset S^2$ such that the $f(x) = f(-x)$.*

Proof. Suppose not. Then we can define a map $g : S^2 \longrightarrow S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

Then g satisfies $g(-x) = -g(x)$. Let $\gamma : S^1 \longrightarrow S^1$ be the restriction to the equator. Note that since γ extends over the northern (or southern) hemisphere, the loop γ is null. We also write δ for the composition $I \longrightarrow S^1 \xrightarrow{\gamma} S^1$.

The equation $g(-z) = -g(z)$ means that $\gamma(-z) = -\gamma(z)$ or $\delta(t + \frac{1}{2}) = -\delta(t)$. Denote by $\tilde{\delta}$ a lift to a path in $\mathbb{R}$. Then $\tilde{\delta}$ must satisfy the equation $\tilde{\delta}(t + \frac{1}{2}) = \tilde{\delta}(t) + \frac{1}{2} + k$ for some integer k. In particular, we find that

$$\tilde{\delta}(1) = \tilde{\delta}\left(\frac{1}{2}\right) + \frac{1}{2} + k = \tilde{\delta}(0) + 1 + 2k.$$

Thus the degree of γ is the odd integer $1 + 2k$. This contradicts that γ is null. ∎

Application: At any point in time, there are two polar opposite points on Earth having the same temperature and same barometric pressure. (Or pick any two continuously varying parameters)

Corollary 1.7. *The sphere S^2 is not homeomorphic to any subspace of $\mathbb{R}^2$.*

Proof. According to the theorem, there is no continuous injection $S^2 \longrightarrow \mathbb{R}^2$. ∎

1.2. **Fundamental group of spheres.** We saw that S^1 has a nontrivial fundamental group, but in contrast we will see that the higher spheres all have trivial fundamental groups. A (path-connected) space with trivial fundamental group is said to be **simply connected**.

Theorem 1.8. *The n-sphere S^n is simply connected if $n \geq 2$.*

This follows from the following theorem.

Theorem 1.9. *Any continuous map $S^1 \longrightarrow S^n$ is path-homotopic to one that is not surjective.*

Let's first use this to deduce the statement about n-spheres. Let γ be a loop in S^n. We know it is path-homotopic to a loop δ that is not surjective. But recall that $S^n - \{P\} \cong \mathbb{R}^n$. Thus we can contract δ using a straight-line homotopy in the complement of any missed point. It remains to prove the latter theorem.

Proof. There are a number of ways to prove this result. For instance, it is an easy consequence of "Sard's Theorem" from differential topology. Here is a proof using once again the Lebesgue number lemma.

Let $\{U, V\}$ be the covering of S^n, where U is the upper (open) hemisphere, and V is the complement of the North pole. Let $\gamma : S^1 \longrightarrow S^n$ be a loop. By Lebesgue, we can subdivide the interval I into finitely many subintervals $[s_i, s_{i+1}]$ such that on each subinterval, γ stays within either U or V. We will deform γ so that it misses the North pole. On the subintervals that are mapped into V, nothing needs to be done.

Suppose $[s_i, s_{i+1}]$ is not mapped into V, so that γ passes through the North pole on this segment. Recall that the open hemisphere U is homeomorphic to $\mathbb{R}^n$. The problem thus reduces to the following: given a path in $\mathbb{R}^n$, show it is path-homotopic to one not passing through the origin.

This is simple. First, any path is homotopic to the straight-line path. If that does not pass through the origin, great. If it does, just wiggle it a little, and it won't any more. ■

Corollary 1.10. *The infinite sphere S^∞ is simply connected.*

Proof. Consider a loop α in S^∞. The image of α is then a compact subset of the CW complex S^∞. It follows (see Hatcher, A.1) that the image of α is contained in a finite union of cells. In other words, the image of α is contained in some S^n. By the above, α is null-homotopic in S^n and therefore in S^∞ as well. ■

Fri, Jan. 19

You showed on your homework that S^∞ is contractible, and this in fact implies simply connected, as the next result shows.

Theorem 1.11. *Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then, for any choice of basepoint $x \in X$, the induced map*

$$f_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(Y, f(x))$$

is an isomorphism.

At first glance, this might seem obvious, since we have a quasi-inverse $g : Y \longrightarrow X$ to f, and so we would expect g_* to be the inverse of f_*. But note that there is no reason that $g(f(x))$ would be x again, so g_* does not even map to the correct group to be the inverse of f_*. We need to employ some sort of change-of-basepoint to deal with this. So we take a little detour to address this issue.

1.3. Dependence on the basepoint.

Although we often talk about "the fundamental group" of a space X, this group depends on the choice of basepoint for the loops. One thing at least should be clear: if we want to understand $\pi_1(X, x_0)$, only the path component of x_0 in X is relevant. Any other path component can be ignored. More precisely, if PC_x denotes the path-component of a point x, then for any choice of basepoint x_0, we get an **isomorphism of groups**

$$\pi_1(PC_{x_0}, x_0) \cong \pi_1(X, x_0).$$

For this reason, we will often assume from now on that our spaces are path-connected.

Under this assumption that X is path-connected, how does the fundamental group depend on the choice of base point? Suppose that x_0 and x_1 are points in X. How can we compare loops based at x_0 to loops based at x_1? Since X is path-connected, we may choose some path α in X from x_0 to x_1. Then we may use the change-of-basepoint technique that we discussed at the end of the fall semester. If γ is a loop based at x_0, we get a loop $\overline{\alpha} \cdot \gamma \cdot \alpha$ based at x_1. Let us write $\Phi_\alpha(\gamma)$ for this loop. The same argument we gave in the case $X = S^1$ generalizes to give

Proposition 1.12.

1. *The operation Φ_α gives a well-defined operation on homotopy-classes of loops.*
2. *The operation Φ_α only depends on the homotopy-class of α.*
3. *The operation Φ_α induces an isomorphism of groups*

$$\Phi_\alpha : \pi_1(X, x_0) \cong \pi_1(X, x_1)$$

with inverse induced by $\Phi_{\overline{\alpha}}$.

So, as long as X is path-connected, the isomorphism-type of the fundamental group of X does not depend on the basepoint. For example, once we know that $\pi_1(\mathbb{R}^2, \mathbf{0}) = \langle e \rangle$, it follows that the same would be true with any other choice of basepoint. More generally, we know that any convex subset of $\mathbb{R}^n$ is simply connected.

Proposition 1.13. *Let h be a homotopy between maps $f, g : X \rightrightarrows Y$. For a chosen basepoint $x_0 \in X$, define a path α in Y by $\alpha(s) = h(x_0, s)$. Then the diagram to the right commutes.*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\ & \searrow^{g_*} & \downarrow \cong \ \Phi_\alpha \\ & & \pi_1(Y, g(x_0)) \end{array}$$

Proof. For any loop γ in X based at x_0, we want a path-homotopy $H : \Phi_\alpha(f \circ \gamma) \simeq_p g \circ \gamma$. For convenience, let us write $y_0 = g(x_0)$. For each t, let α_t denote the path $\alpha_t(s) = \alpha(1 - (1 - s)t)$. Note that $\alpha_1 = \alpha$ and α_0 is constant at $\alpha(1) = y_0$.

Then the function

$$H_t = \overline{\alpha_t} \cdot (h_t \circ \gamma) \cdot \alpha_t$$

defines a path-homotopy $\overline{c}_{y_0} \cdot g(\gamma) \cdot c_{y_0} \simeq_p \overline{\alpha} \cdot f(\gamma) \cdot \alpha = \Phi_\alpha(f(\gamma))$. ■

Proof of Theorem 1.11. Let $g : Y \longrightarrow X$ be a quasi-inverse to f. Then $g \circ f \simeq \mathrm{id}_X$, so Prop 1.13 gives us a diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow[\cong]{\mathrm{id}_*} & \pi_1(X, x_0) \\ & \searrow^{(gf)_*} & \downarrow \cong \ \Phi_\alpha \\ & & \pi_1(X, gf(x_0)) \end{array}$$

Now $(gf)_*$ must be an isomorphism since the other two maps in the diagram are isomorphisms. Since $(gf)_* = g_* \circ f_*$, the map f_* must be injective and similarly g_* must be surjective.

But now we can swap the roles of f and g, getting a diagram

$$\begin{array}{ccc} \pi_1(Y, f(x_0)) & \xrightarrow[\cong]{\mathrm{id}_*} & \pi_1(Y, f(x_0)) \\ & \searrow^{(fg)_*} & \downarrow \cong \ \Psi_\alpha \\ & & \pi_1(Y, fgf(x_0)) \end{array}$$

It then follows that $g_* : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, gf(x_0))$ is injective. Since we already showed it is surjective, we deduce that it is an isomorphism. Now going back to our first diagram, we get

$$g_* \circ f_* = \Phi_\alpha, \qquad \text{or} \qquad f_* = g_*^{-1} \circ \Phi_\alpha,$$

so that $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism. ■

So far, we know a number of simply connected spaces ($\mathbb{R}^n$, S^n for $n \geq 2$), and we know that $\pi_1(T^n) \cong \mathbb{Z}^n$ for any $n \geq 1$. Can there be torsion in the fundamental group? For example, is it possible that for some nontrivial loop γ in X, winding around the loop twice gives a trivial loop? The next example will have this property.