Mon, Jan. 22

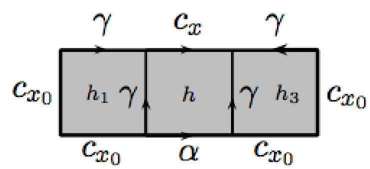
1.4. **Fundamental group of $\mathbb{RP}^2$.** Recall that the real projective plane $\mathbb{RP}^2$ is defined as the quotient of S^2 by the equivalence relation $x \sim -x$. The equivalence classes are precisely the sets of pairs of antipodal points. Another way to think about this is that each pair of antipodal points corresponds to a straight line through the origin. We will determine $\pi_1(\mathbb{RP}^2)$. Today, we're going to calculate $\pi_1(\mathbb{RP}^2)$, but first I want to discuss a result about contractibility of paths.

Proposition 1.14. (1) *Let $\alpha \in \pi_1(X, x_0)$. Then $\alpha \simeq_p c_{x_0}$ if and only if $\alpha : S^1 \longrightarrow X$ extends to a map $D^2 \longrightarrow X$.*

(2) *Let α and β be paths in X from x to y. Then $\alpha \simeq_p \beta$ if and only if the loop $\alpha * \overline{\beta}$ is null.*

Proof.

(1) ($\Rightarrow$) This follows from Homework II.1.
($\Leftarrow$) Again using Homework II.1, we may assume given a homotopy $h : \alpha \simeq c_x$. Since h is not assumed to be a path-homotopy, the formula $\gamma(s) = h(0, s)$ defines a possible nontrivai path. The picture



where $h_1(s,t) = \gamma(st)$ and $h_3(s,t) = \overline{\gamma}(st)$, defines a path-homotopy $H : \alpha \simeq_p \gamma \cdot c_x \cdot \overline{\gamma}$.

(2) The point is that

$$\alpha \simeq_p \beta \qquad \Rightarrow \qquad \alpha\overline{\beta} \simeq_p \beta\overline{\beta} \simeq_p c_x$$

and similarly

$$\alpha\overline{\beta} \simeq_p c_x \qquad \Rightarrow \qquad \alpha \simeq_p \alpha\overline{\beta}\beta \simeq_p c_x\beta \simeq_p \beta$$

■

Recall that for S^1, the exponential map $p : \mathbb{R} \longrightarrow S^1$ was key. The analogue of that map for $\mathbb{RP}^2$ will be the quotient map

$$q : S^2 \longrightarrow \mathbb{RP}^2.$$

Note that in this case, the "fiber" (the preimage of the basepoint) consists of two points. Another ingredient that was used for S^1 was that it has a nice cover. The same is true for $\mathbb{RP}^2$: there is a cover of $\mathbb{RP}^2$ by open sets U_1, U_2, U_3 such that each preimage $q^{-1}(U_i)$ is a disjoint union $V_{i,1} \amalg V_{i,2}$ such that on each component $V_{i,j}$, the map q gives a homeomorphism $q : V_{i,j} \cong U_i$. For instance, U_1 consists of points $q(x,y,z)$ with $x \neq 0$. Then $q^{-1}(U_1)$ is the disjoint union of the left and right open hemispheres in S^2. On each hemisphere H, q restricts to a homeomorphism $q : H \cong U_1$.

For any point $x \in q^{-1}(\overline{1}) = \{-1, 1\}$, we define a loop $\Gamma(x)$ at $\overline{1}$ in $\mathbb{RP}^2$ as follows: take any path α in S^2 from 1 to x. Then $\Gamma(x) = q\alpha$ is a loop in $\mathbb{RP}^2$. Note that this is well-defined **because S^2 is simply-connected**, so that any two paths between 1 and x are homotopic. When $x = 1$, this of course gives the class of the constant loop, but when $x = -1$, this gives a nontrivial loop in $\mathbb{RP}^2$. We claim that this is a bijection. So there is only one nontrivial loop!

To see this, we construct an inverse $w : \pi_1(\mathbb{RP}^2) \longrightarrow \{-1, 1\}$. We need some lemmas:

Lemma 1.15. *Given any loop in $\mathbb{RP}^2$, there is a unique lift to a path in S^2 starting at 1.*

The proof of this lemma is **exactly the same** as that of the first lemma in the proof for the circle.

Lemma 1.16. *Let $h : \gamma \simeq_p \delta$ be a path-homotopy between loops at $\overline{1}$ in $\mathbb{RP}^2$. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow S^2$ such that $\tilde{h}(0,0) = 1$.*

Again, the proof here is identical to that for the sphere. Let's see how we can use the lemmas to define w. Given any loop γ in $\mathbb{RP}^2$, there is a unique lift $\tilde{\gamma}$ in S^2 starting at 1. Since it is a lift of a loop, we must have $\tilde{\gamma}(1) \in \{-1, 1\}$. So we define $w(\gamma) = \tilde{\gamma}(1)$. That this is well-defined follows from the second lemma.

It remains to show that w really is the inverse. Let $x \in \{-1, 1\}$. Then $\Gamma(x) = q \circ \alpha$ for some path α in S^2 from 1 to x. To compute $w(\Gamma(x))$, we must find a lift of $\Gamma(x)$, but we already know that α is the lift. Thus $w(\Gamma(x)) = \alpha(1) = x$.

Similarly, suppose γ is any loop in $\mathbb{RP}^2$. Let $\tilde{\gamma}$ be a lift. Then $\Gamma(w(\gamma)) = \Gamma(\tilde{\gamma}(1)) = q\alpha$, where α is any path from 1 to $\tilde{\gamma}(1)$. But of course $\tilde{\gamma}$ is such a path and $\gamma = q\tilde{\gamma}$.

Note that we have given a bijection between $\pi_1(\mathbb{RP}^2)$ and $\{-1, 1\}$, but we have not talked about a group structure. That's because we don't need to: there is only one group of order two! We have shown that

$$\pi_1(\mathbb{RP}^2) \cong C_2.$$

In fact, the same proof (replacing S^2 by S^n) shows that, for $n \geq 2$, we have $\pi_1(\mathbb{RP}^n) \cong C_2$.

Wed, Jan. 24

1.5. **Fundamental group of $S^1 \vee S^1$.** We will do one more example before describing the repeated phenomena we have seen in these examples. First, recall from last semester that given based spaces (X, x_0) and (Y, y_0), their **wedge sum**, or one-point union, is $X \vee Y = X \amalg Y/\sim$, where $x_0 \sim y_0$. Today, we want to study the fundamental group of $S^1 \vee S^1$ following the same approach as in the previous examples. We want to once again find a nice map $p : X \longrightarrow S^1 \vee S^1$ for some X. What we really want is an example of the following:

Definition 1.17. A surjective map $p : E \longrightarrow B$ is called a **covering map** if every $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union $p^{-1}(U) = \amalg_i V_i$ and such that p restricts to a homeomorphism $p : V_i \xrightarrow{\cong} U$. We say that the neighborhood U is **evenly covered** by p.

Remark 1.18. It is common to assume that E is connected and locally path-connected. We will assume this from now on, as it simplifies the theory. So as to avoid repeatedly saying (or writing) "connected and locally path-connected", I will simply call these spaces **very connected**.

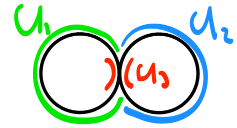
It is important to note that the neighborhood condition is local in B, not E. This contrasts with the following definition.

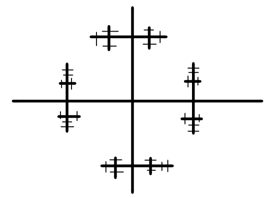
Definition 1.19. A map $f : X \longrightarrow Y$ is said to be a **local homeomorphism** if every $x \in X$ has a neighborhood U such that $f(U) \subseteq Y$ is open and $f_{|U} : U \xrightarrow{\cong} f(U)$ is a homeomorphism.

Every covering map is a local homeomorphism: given $e \in E$, take an evenly covered neighborhood U of $p(e)$. Then e is contained in one of the V_j's, which is the desired neighborhood . The converse is not true, as the next example shows.

Example 1.20. Consider the usual exponential map $p : \mathbb{R} \longrightarrow S^1$, but now restrict it to $(0, 8.123876)$. This is a local homeomorphism but not a covering map. For instance, the standard basepoint of S^1 has no evenly covered neighborhood under this map.

Ok, now back to $S^1 \vee S^1$. It is tempting to take $X = \mathbb{R}$ since $S^1 \vee S^1$ looks locally like a line, but there is a problem spot at the crossing of the figure eight. To fix this, we might try to take X to be the union of the coordinate axes inside of $\mathbb{R}^2$. This space is really just $\mathbb{R} \vee \mathbb{R}$, and so we have the map $p \vee p : \mathbb{R} \vee \mathbb{R} \longrightarrow S^1 \vee S^1$. We want a cover of $S^1 \vee S^1$ which is nicely compatible with our map from X. Suppose we consider the cover U_1, U_2, and U_3, where U_1 is the complement of the basepoint in one circle, U_2 is the complement of the basepoint in the other, and finally U_3 is some small neighborhood of the basepoint. Well, U_1 and U_2 are good neighborhoods for $p \vee p$, but U_3 is not. The map $p \vee p$ does not give a homeomorphism from each component of the preimage of U_3 to U_3. To fix this, we would want to add infinitely many cross-sections to each of the axes.





Instead, we take X to be the fractal space given in the picture (see also page 59 of Hatcher). We define $p : X \longrightarrow S^1 \vee S^1$ as follows. On *horizontal* segments, use the exponential map to the *right* branch of $S^1 \vee S^1$. On *vertical* segments, use the left branch. Then the cover U_1, U_2, and U_3 from above is compatible with this new map p, and we see that p is a covering map.

Lemma 1.21. *The space X is simply-connected.*

Proof. The main point is that any loop in X is compact and therefore contained in a *finite* union of edges. Consider the edge furthest from the basepoint that contains part of the loop. The loop is homotopic to one constant on this furthest edge. This furthest edge is now no longer needed, and we have a new furthest edge. We can repeat until the loop is completely contracted. ■

Let $F = p^{-1}(*)$ be the fiber. Any point in this fiber may be uniquely described as a "word" in the letters u, r, d, and l. Define

$$\Gamma : F \longrightarrow \pi_1(S^1 \vee S^1)$$

as follows: given $y \in F$, let α_y be any path in X from the basepoint to y. Then $\Gamma(y) = p \circ \alpha$. It does not matter which α_y we choose since X is simply-connected. We will define an inverse to Γ, but we now state the needed lemmas in the generality of coverings.

Lemma 1.22. *Let $p : E \longrightarrow B$ be a covering and suppose $p(e) = b$. Given any path starting at b in B, there is a unique lift to a path in E starting at e.*

The proof of this lemma is **exactly the same** as that of Lemma 1.3, for the circle.

Lemma 1.23. *Let $p : E \longrightarrow B$ be a covering and suppose $p(e) = b$. Let $h : \gamma \simeq_p \delta$ be a path-homotopy between paths starting at b in B. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow E$ such that $\tilde{h}(0,0) = e$.*

Just as in the previous examples, the above lemmas allow us to define $w : \pi_1(S^1 \vee S^1) \longrightarrow F$ by the formula $w(\gamma) = \tilde{\gamma}(1)$. We will skip the verification that Γ and w are inverse, as this really follows the same script.

Fri, Jan. 26

We have established a bijection between $\pi_1(S^1 \vee S^1)$ and the set of "words" in the letters u, r, d, and l. It remains to describe the group structure. For this, we will back up a little.

Definition 1.24. Let $p : E \longrightarrow B$ and $q : E' \longrightarrow B$ be covers of a space B. A **map of covers** from E to E' is simply a map of spaces $\varphi : E \longrightarrow E'$ such that $q \circ f = p$. These are also sometimes called **covering homomorphisms**.

The special case in which the two covers are the *same* cover and f is a homeomorphism is referred to as a **deck transformation**. We write $\mathrm{Aut}(E)$ for the set of all deck transformations of E. This is a group under composition.

Keeping our notation from earlier, let $b \in B$ be a basepoint and write $F = p^{-1}(b)$ for the fiber. Note that any deck transformation $\varphi : E \longrightarrow E$ must take F to F. Let us pick a basepoint e for E. Since we want the covering map q to be based, this means that e lies in the fiber F. We may now define a map $A : \mathrm{Aut}(E) \longrightarrow F$ by $A(\varphi) = \varphi(e)$.

Theorem 1.25. *Let $p : X \longrightarrow B$ be a covering such that X is simply connected. Then the map $A : \mathrm{Aut}(X) \longrightarrow F$ is a bijection and the composition $\Gamma \circ A$ is an isomorphism of groups $\mathrm{Aut}(X) \cong \pi_1(B)$.*

Proof. Let us first show that A is injective. Thus let φ_1 and φ_2 be deck transformations which agree at e. Let $x \in X$ be any point and let α be any path in X from e to x. Then the paths $\varphi_1 \circ \alpha$ and $\varphi_2 \circ \alpha$ are both lifts of $p \circ \alpha$ starting at the common point $\varphi_1(e) = \varphi_2(e)$. By the uniqueness of lifts, these must be the same path. It follows that their endpoints, $\varphi_1(x)$ and $\varphi_2(x)$ agree.

It remains to show that A is surjective. Let $f \in F$ be any point in the fiber. We wish to produce a deck transformation $\varphi : X \longrightarrow X$ such that $\varphi(e) = f$. We build the map φ locally and patch together. Let $x \in X$ and pick any path $\alpha : e \rightsquigarrow x$. Then $p\alpha$ is a path in B starting at b and ending at px. By the path-lifting lemma, there is a unique lift $\widetilde{p\alpha}$ in X starting at f. We define $\varphi(x) = \widetilde{p\alpha}(1)$. From this definition, continuity is not at all clear. But the point is that since p is a covering, we can choose an evenly-covered neighborhood U of $p(x)$. Let V be the slice of $p^{-1}(U)$ containing x and V' the slice containing $\varphi(x) = \widetilde{p\alpha}(1)$. Then the restriction of φ to V is the composition of homeomorphisms

$$V \xrightarrow{p} U \xleftarrow{p} V'.$$

By the local criterion for continuity (Prop 2.19 in Lee), it follows that φ is continuous.

By construction, φ will be a map of covers, as long as we can verify that it is well-defined. But if $\delta : e \rightsquigarrow x$ is another choice of path, we know that $\alpha \simeq_p \delta$ because X is simply-connected. It follows that $p\alpha \simeq_p p\delta$, and by lifting the path-homotopy, it follows that $\widetilde{p\alpha} \simeq_p \widetilde{p\delta}$, so that their right endpoints agree.

We will finish the proof next time.