Mon, Jan. 29

So given $f \in F$, we have built a map of covers $\varphi : X \longrightarrow X$, but we wanted this to be an isomorphism. From the construction of φ, we see that it is a local homeomorphism, which implies that it is open. Suppose $\varphi(x_1) = \varphi(x_2)$. Note that since φ is a map of covers, this implies that x_1 and x_2 are in the same fiber. Let $\alpha_1 : e \rightsquigarrow x_1$ and $\alpha_2 : e \rightsquigarrow x_2$ be paths. By hypothesis, $\widetilde{p\alpha_1}$ and $\widetilde{p\alpha_2}$ have the same endpoints. Since X is simply-connected, we know that $\widetilde{p\alpha_1} \simeq_p \widetilde{p\alpha_2}$. It follows that $p\alpha_1 \simeq_p p\alpha_2$, and it then follows, by lifting the homotopy, that $\alpha_1 \simeq_p \alpha_2$. In particular, $\alpha_1(1) = \alpha_2(1)$, so $x_1 = x_2$. This shows that φ is injective.

To see that φ is surjective, let $x \in X$. We can then pick a path $\gamma : f \rightsquigarrow x$. Then $p\gamma$ is a path in B from b to $p(x)$, which lifts uniquely to a path $\tilde{\gamma}$ from e to some point x'. But then $\varphi(x') = x$ by the definition of φ.

We have now established that

$$A : \mathrm{Aut}(X) \longrightarrow F$$

is a bijection. We also wanted to show that the resulting bijection $\Gamma \circ A : \mathrm{Aut}(X) \longrightarrow \pi_1(B)$ is a group isomorphism. It remains only to show that this is a group homomorphism.

Let $\varphi_1, \varphi_2 \in \mathrm{Aut}(X)$. Recall that $\Gamma(A(\varphi_1))$ is defined as follows: pick any path α_1 in X from e to $f_1 = \varphi_1(e)$. Then $\Gamma(A(\varphi_1)) = p \circ \alpha_1$. Similarly $\Gamma(A(\varphi_2)) = p \circ \alpha_2$. Now $A(\varphi_2 \circ \varphi_1) = \varphi_2 \circ \varphi_1(e) = \varphi_2(f_1)$. To compute Γ of this point, we need a path in X from e to $\varphi_2(f_1)$. But $\alpha_2 * \varphi_2(\alpha_1)$ is such a path. Then

$$\begin{aligned} \Gamma(A(\varphi_2 \circ \varphi_1)) = \Gamma(\varphi_2(f_1)) &= p \circ (\alpha_2 * \varphi_2(\alpha_1)) = (p \circ \alpha_2) * (p \circ \varphi_2 \circ \alpha_1) \\ &= (p \circ \alpha_2) * (p \circ \alpha_1) = \Gamma(A(\varphi_2)) * \Gamma(A(\varphi_1)). \end{aligned}$$

■

Returning now to our example $X \longrightarrow S^1 \vee S^1$, we have identified $\pi_1(S^1 \vee S^1)$ with the group of deck transformations $X \cong X$, and we know we have one such deck transformation for each point in the fiber F. Any transformation can be thought of as a sequence of horizontal and vertical "moves". Writing u for an upwards shift and r for a shift to the right, any element of the group can be described by a seqence of u's, r's, and their inverses.

Definition 1.26. A **word** in letters u, r, and their inverses is simply a sequence of these letters. We say the word is **reduced** if no u^{-1} is adjacent to a u, and similarly for the r's. The **free group** F_2 or $F(u, r)$ on the letters u and r is the set of reduced (including empty) words, where the group operation is concatenation. The inverse of any word is the same word in reversed order and with the sign of each letter reversed.

We have shown that $\pi_1(S^1 \vee S^1)$ is the free group on two letters. In particular, this is our first example of a nonabelian fundamental group.

Wed, Jan. 31

1.6. **The theory of covering spaces.** So far, the only kind of coverings we have studied have been those in which the covering space is simply connected. Now we will relax this condition and discuss the more general theory.

Proposition 1.27. *Let $p : E \longrightarrow B$ be a covering. Then the induced map $p_* : \pi_1(E) \longrightarrow \pi_1(B)$ is injective.*

Proof. Let $\gamma \in \pi_1(E)$ and suppose $p_*(\gamma) = 0$. In other words, the loop $p \circ \gamma$ in B is null. Let $h : I \times I \longrightarrow B$ be a null-homotopy. Then this lifts to a homotopy $\tilde{h} : I \times I \longrightarrow E$ from γ (the unique lift of $p \circ \gamma$) to a lift $\tilde{c}$ of the constant loop. Since the constant loop at e is a lift of the constant loop at b, uniqueness of lifts implies that $\tilde{c}$ is the constant loop. So $\tilde{h}$ is a null-homotopy for γ. ■

Example 1.28. The only example of a covering we have discussed thus far in which the covering space is not simply connected is the n-fold cover $S^1 \longrightarrow S^1$. In this case, the cover sends the generator of $\pi_1(S^1) \cong \mathbb{Z}$ to n times the generator, and the image of p_* is the subgroup $n\mathbb{Z} < \mathbb{Z}$.

Given the above result, any covering of B gives rise to a subgroup of $\pi_1(B)$. One might wonder what subgroups can arise in this way. We will see that, under mild hypotheses on B, every subgroup arises in this way.

Previously, we have studied lifting paths and path-homotopies against a covering. We can also generalize this to consider lifting arbitrary maps $f : Z \longrightarrow B$. As in Remark 1.18, whenever we discuss a covering map $E \longrightarrow B$, we assume that E is "very connected", which implies the same for B. In particular, this is assumed for the following results.

Proposition 1.29. *(Homotopy lifting) Let Z be a locally connected space. Let $p : E \longrightarrow B$ be a covering and $h : Z \times I \longrightarrow B$ be a homotopy between maps $f, g : Z \rightrightarrows B$. Let $\tilde{f}$ be a lift of f. Then there is a unique lift of h to $\tilde{h}$ with $\tilde{h}_0 = \tilde{f}$.*

Proposition 1.30. *(Unique lifting) Let $p : E \longrightarrow B$ be a covering and $f : Z \longrightarrow B$ a map, with Z connected. If $\tilde{f}$ and $\hat{f}$ are both lifts of f that agree at some point of Z, then they are the same lift.*

Note that in the second result, we are not asserting that a lift exists! See Theorems 8.3 and 8.4 of [Lee] for complete proofs.

Here is a sketch of Proposition 1.30.

Sketch. The idea is to show that the subset of Z on which the lifts agree is both open and closed; it is already given to be nonempty. For any $z \in Z$, pick an evenly-covered neighborhood U of $f(z)$. On the one hand, suppose $\tilde{f}(z) = \hat{f}(z)$. Then let V be the component of $p^{-1}(U)$ containing this point. Then $\tilde{f}^{-1}(V) \cap \hat{f}^{-1}(V)$ is a neighborhood of z on which the lifts agree (since $q : V \longrightarrow U$ is a homeomorphism).

On the other hand, if $\tilde{f}(z) \neq \hat{f}(z)$, then let $\tilde{V}$ and $\hat{V}$ be the components of $\tilde{f}(z)$ and $\hat{f}(z)$ in $p^{-1}(U)$. It follows that $\tilde{f}^{-1}(\tilde{V}) \cap \hat{f}^{-1}(\hat{V})$ is a neighborhood of z on which $\tilde{f}$ and $\hat{f}$ disagree (they land in different components of $p^{-1}(U)$). ■

Fri, Feb. 2

The interesting, new result here concerns the existence of lifts.

Proposition 1.31. *(Lifting Criterion) Let $p : E \longrightarrow B$ be a covering and let $f : Z \longrightarrow B$, with Z very connected. Given points $z_0 \in Z$ and $e_0 \in E$ with $f(z_0) = p(e_0)$, there is a lift $\tilde{f}$ with $\tilde{f}(z_0) = e_0$ if and only if $f_*(\pi_1(Z, z_0)) \subseteq p_*(\pi_1(E, e_0))$.*

Proof. ($\Rightarrow$) Since $f = p \circ \tilde{f}$, we have $f_* = p_* \circ \tilde{f}_*$.

($\Leftarrow$) Here is the more interesting direction. Suppose that $f_*(\pi_1(Z, z_0)) \subseteq p_*(\pi_1(E, e_0))$. Let $z \in Z$. We wish to define $\tilde{f}(z)$. Pick any path α in Z from z_0 to z. Then $f \circ \alpha$ is a path in B, which therefore lifts uniquely to a path $\tilde{\alpha}$ in E starting at, say e_0. We define $\tilde{f}(z) = \tilde{\alpha}(1)$. Then $\tilde{f}$ is a lift of f.

Why is the lift $\tilde{f}$ well-defined? Suppose β is another path in Z from z_0 to z. Then $f \circ (\alpha \cdot \overline{\beta})$ is a loop in B at $b_0 = f(z_0)$. By assumption, this means that for some loop δ in E, we have

$$p \circ \delta \simeq_p f \circ (\alpha \cdot \overline{\beta}) = f(\alpha) \cdot \overline{f(\beta)}$$

in B. Since path-composition behaves well with respect to path-homotopy, we have a path-homotopy

$$h : (p \circ \delta) \cdot f(\beta) \simeq_p f(\alpha)$$

of paths in B. Note that the path $(p \circ \delta) \cdot f(\beta)$ lifts to the path $\delta \cdot \tilde{\beta}$. The homotopy h then lifts (uniquely) to a path-homotopy in E

$$\tilde{h} : \delta \cdot \tilde{\beta} \simeq_p \tilde{\alpha}.$$

In particular, these have the same endpoints. Of course, the endpoint of $\delta \cdot \tilde{\beta}$ is simply the endpoint of $\tilde{\beta}$. It follows that $\tilde{f}$ is well-defined at z.

Just for emphasis, let's go through the proof that $\tilde{f}$ is continuous. Let $z \in Z$ and let U be an evenly covered neighborhood U of $f(z)$, and let V be the component of $p^{-1}(U)$ containing the lift $\tilde{f}(z)$. Let $W \subseteq Z$ be the path-component of $f^{-1}(U)$ containing z. Since Z is locally path-connected, W is open. Moreover, since W is path-connected and $\tilde{f}(W) \cap V \neq \emptyset$, we must have $\tilde{f}(W) \subseteq V$. Then on the neighborhood W of z, the lift $\tilde{f}$ may be described as the composition $p|_V^{-1} \circ f$. It follows that $\tilde{f}$ is continuous on the neighborhood W of z. Since z was arbitrary, $\tilde{f}$ is continuous. ■

This implies what we already know: S^1 is not a retract of $\mathbb{R}$. More generally, and less trivially, we have that the identity map $S^1 \longrightarrow S^1$ does not lift against the n-fold cover $p_n : S^1 \longrightarrow S^1$. Even more generally, we might ask about lifting some $p_k : S^1 \longrightarrow S^1$ against the cover $p_n : S^1 \longrightarrow S^1$. By the result above, this happens if and only if $k\mathbb{Z} \subseteq n\mathbb{Z}$. In other words, this happens if and only if $n \mid k$.

More interestingly, we have

Corollary 1.32. *Suppose that the covering space E is simply-connected. Then a map $f : Z \longrightarrow B$ lifts to some $\tilde{f} : Z \longrightarrow E$ if and only if f induces the trivial map on fundamental groups.*

Corollary 1.33. *Suppose that Z is simply-connected and $p : E \longrightarrow B$ is a covering map. Then any map $f : Z \longrightarrow B$ lifts against p.*

Thus if $X \longrightarrow B$ is a simply connected covering and $E \longrightarrow B$ is any covering, we automatically get a map of covers $X \longrightarrow E$. For this reason, simply connected covers are referred to as **universal covers**.