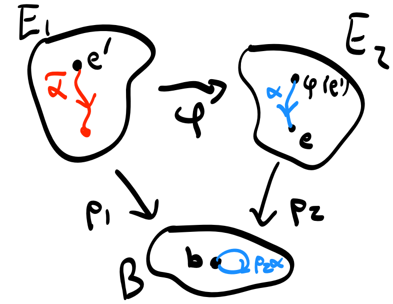
Mon, Feb. 5

Proposition 2.8. *Suppose that $\varphi : E_1 \longrightarrow E_2$ is a map of covers. Then φ is a covering map.*

Proof. We start by showing that φ is surjective. Let $e \in E_2$. Let $b = p_2(e)$, and pick any $e' \in p_1^{-1}(b)$. Since E_2 is very connected, we can find a path $\alpha : \varphi(e') \rightsquigarrow e$ in E_2. We can push this path α down to a loop $p_2\alpha$ in B and then lift this uniquely to a path $\tilde{\alpha}$ in E_1 starting at e'. Now $\varphi(\tilde{\alpha})$ is a lift of $p_2\alpha$ in E_2 starting at $\varphi(e')$, so by uniqueness of lifts, we must have $\varphi(\tilde{\alpha}) = \alpha$. In particular, $\varphi(\tilde{\alpha}(1)) = e$.

Now we show that e has an evenly-covered neighborhood of e. We know that the point $p_2(e) \in B$ has an evenly covered neighborhood U_2 (with respect to p_2). Let U_1 be an evenly covered neighborhood, with respect to p_1, of $p_2(e)$. Write U for the component of $U_1 \cap U_2$ containing $p_2(e)$. Then $p_2^{-1}(U) \cong \amalg V_i$. Let V_0 be the component containing e. Write $p_1^{-1}(U) \cong \amalg W_j$. Then, since U is connected, each V_i and W_j must be connected. It follows that φ takes each W_j into a single V_i, so that $\varphi^{-1}(V_0) \subseteq p_1^{-1}(U)$ is a disjoint union of some of the W_j's, and it follows that φ restricts to a homeomorphism on each component because both p_1 and p_2 do so.



■

It follows that any universal cover $X \longrightarrow B$ *covers every other covering* $E \longrightarrow B$.

Remark 2.9. Recall that in the proof of Theorem 1.25, we ended up building a map of covers $\varphi : X \longrightarrow X$ corresponding to any point in the fiber F, but we wanted to know it was in fact a homeomorphism. Prop 2.8 now gives us that it is a covering map, so that according to the homework, it suffices to show that the φ we constructed was injective. This can be seen by verifying that it is injective on each fiber.

2.2. **The monodromy action.** Our next goal is to completely understand the possible covers of a given space B. There are two avenues of approach. On the one hand, Prop. 2.1 tells us that covering spaces give rise to subgroups of $\pi_1(B)$, so we can try to understand the collection of subgroups. Another approach, which we will look at next, focuses on the fiber $F = p^{-1}(b_0)$.

It will be convenient in what follows to write $G = \pi_1(B, b_0)$ and $F = p^{-1}(b_0) \subset E$. Given a loop γ based at b_0 and a point $f \in F$, we will write $\tilde{\gamma}_f$ for the lift of γ which starts at f.

Theorem 2.10. *Let $p : E \longrightarrow B$ be a covering and let $F = p^{-1}(b)$ be the fiber over the basepoint. Then the function*

$$a : F \times \pi_1(B) \longrightarrow F, \qquad (f, [\gamma]) \mapsto \tilde{\gamma}_f(1)$$

specifies a transitive right action of $\pi_1(B)$ on the fiber F. This is called the **monodromy action**.

Proof. Recall that we have already showed this to be well-defined.

Let c_{b_0} be the constant loop at b_0. Then the constant loop c_f at f in E is a lift of c_{b_0} starting at f, so by uniqueness it must be the only lift. Thus $f \cdot [c_{b_0}] = f$.

Now let α and β be loops at b. We wish to show that $(f \cdot \alpha) \cdot \beta = f \cdot (\alpha \cdot \beta)$. Let $f_2 = \tilde{\alpha}_f(1)$. Then $\tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}$ is a (= the) lift of $\alpha \cdot \beta$ starting at f, so

$$f \cdot (\alpha \cdot \beta) = \tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}(1).$$

On the other hand, $f \cdot \alpha = \tilde{\alpha}_f(1) = f_2$, so

$$(f \cdot \alpha) \cdot \beta = f_2 \cdot \beta = \tilde{\beta}_{f_2}(1)$$

Finally, to see that this action is transitive, let f_1 and f_2 be points in the fiber F. Let γ be a path in E from f_1 to f_2. Then $\alpha = p \circ \gamma$ is a loop at b_0. Furthermore $\tilde{\alpha}_{f_1} = \gamma$, so $f_1 \cdot \alpha = \gamma(1) = f_2$. ■

Note that if we instead wrote path-composition in the "correct" order (i.e. in the same order as function composition), this would give a left action of $\pi_1(B)$ on F.

By the Orbit-Stabilizer theorem, since G acts transitively on F, there is an isomorphism of right G-sets $F \cong G_{e_0}\backslash G$, where $G_{e_0} \leq G$ is the stabilizer of e_0.

Wed, Feb. 7

Proposition 2.11. *The stabilizer of $e \in F$ under the monodromy action is the subgroup $p_*(\pi_1(E, e)) \leq \pi_1(B, b_0)$.*

Proof. Let $[\gamma] \in \pi_1(E, e)$. Then γ is a lift of $p \circ \gamma$ starting at e, so $e \cdot p_*(\gamma) = \gamma(1) = e$. Thus $p_*(\gamma)$ stabilizes e.

On the other hand, let $[\alpha] \in \pi_1(B, b_0)$ and suppose that $e \cdot [\alpha] = e$. This means that α lifts to a loop $\tilde{\alpha}$ in E. Thus $\alpha = p \circ \tilde{\alpha}$ and $[\alpha] \in p_*(\pi_1(E, e))$. ■

Corollary 2.12. *Let $p : E \longrightarrow B$ be a covering. Then, writing $H = p_*(\pi_1(E, e))$ the map*

$$H\backslash\pi_1(B, b) \xrightarrow{\cong} F.$$
$$H\,\gamma \mapsto f \cdot \gamma$$

is an identification of right $\pi_1(B)$-sets

We have seen that any covering gives rise to a transitive G-set. We would also like to understand maps of coverings.

Definition 2.13. Let X and Y be (right) G-sets. A function $f : X \longrightarrow Y$ is said to be **G-equivariant** (or a map of G-sets) if $f(xg) = f(x) \cdot g$ for all x.

Proposition 2.14. *Let $\varphi : E_1 \longrightarrow E_2$ be a map of covers of B. The induced map on fibers $F_1 \longrightarrow F_2$ is $\pi_1(B)$-equivariant.*

Proof. Let $[\gamma] \in \pi_1(B)$ and $f \in F_1$. We have $f \cdot [\gamma] = \tilde{\gamma}_f(1)$, where $\tilde{\gamma}_f$ is the lift of γ starting at f. Similarly, we have $\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1)$. But $\varphi(\tilde{\gamma})$ is a lift of γ starting at $\varphi(\gamma(0)) = \varphi(f)$, so $\tilde{\gamma}_{\varphi(f)} = \varphi(\tilde{\gamma}_f)$. Thus

$$\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1) = \varphi(\tilde{\gamma}_f)(1) = \varphi(\tilde{\gamma}_f(1)) = \varphi(f \cdot [\gamma]).$$

■

Proposition 2.15. *Let $H, K \leq G$. Then every G-equivariant map $\varphi : H\backslash G \longrightarrow K\backslash G$ is of the form $Hg \mapsto K\gamma g$ for some $\gamma \in G$ satisfying $\gamma H \gamma^{-1} \leq K$.*

Proof. Since $H\backslash G$ is a transitive G-set, an equivariant map out of it is determined by the value at any point. Suppose we stipulate

$$He \mapsto K\gamma.$$

Then equivariance would force

$$Hg \mapsto K\gamma g.$$

Is this well-defined? Since $Hg = Hhg$ for any $h \in H$, we would need $K\gamma g = K\gamma hg$. Multiplying by $g^{-1}\gamma^{-1}$ gives $K = K\gamma h\gamma^{-1}$. Since $h \in H$ is arbitrary, this says that $\gamma H\gamma^{-1} \leq K$. ■

Corollary 2.16. *A G-equivariant map $\varphi : H\backslash G \longrightarrow K\backslash G$ exists if and only if H is conjugate in G to a subgroup of K. The two orbits are isomorphic (as right G-sets) if and only if H is conjugate to K.*

Notation. Given covers (E_1, p_1) and (E_2, p_2) of B, we denote by $\mathrm{Map}_B(E_1, E_2)$ the set of covering homomorphisms $\varphi : E_1 \longrightarrow E_2$. Given two right G-sets X and Y, we denote by $\mathrm{Hom}_G(X, Y)$ the set of G-equivariant maps $X \longrightarrow Y$.

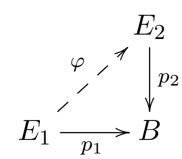
The following theorem classifies covering homomorphisms.

Theorem 2.17. *Let E_1 and E_2 be coverings of B. Then Proposition 2.14 induces a bijection*

$$\mathrm{Map}_B(E_1, E_2) \xrightarrow{\cong} \mathrm{Hom}_G(F_1, F_2).$$

Proof. The key is that a covering homomorphism is a lift in the diagram to the right. Uniqueness of lifts gives injectivity in the theorem. For surjectivity, we use the lifting criterion Prop 2.5. Thus suppose given a G-equivariant map $\lambda : F_1 \longrightarrow F_2$ and fix a point $e_1 \in F_1$. Let $e_2 = \lambda(e_1) \in F_2$. The lifting criterion will provide a lift if we can verify that

$$(p_1)_*(\pi_1(E_1, e_1)) \leq (p_2)_*(\pi_1(E_2, e_2)).$$



But remember that according to Prop 2.11, these are precisely the stabilizers of e_1 and e_2, respectively. Writing H_1 and H_2 for these groups, the map $\lambda : F_1 \longrightarrow F_2$ corresponds to a map

$$\widehat{\lambda} : H_1\backslash G \longrightarrow H_2\backslash G.$$

According to Prop 2.15, this means that $\gamma H_1 \gamma^{-1} \leq H_2$, where $\widehat{\lambda}(H_1 e) = H_2\gamma$. The fact that $\lambda(e_1) = e_2$ means that $\gamma = e$. So $H_1 \leq H_2$ as desired. ■

Corollary 2.18. *If E is a cover of B, then we have group isomorphisms*

$$\mathrm{Aut}_B(E) \cong \mathrm{Aut}_G(H\backslash G, H\backslash G) \cong N_G(H)/H,$$

where $N_G(H)$ is the **normalizer** *of H in G, consisting of those elements of G which conjugate H to itself.*

Proof. Theorem 2.17 gives the first bijection. By Corollary 2.15, we have a surjective group homomorphism $N_G(H) \longrightarrow \mathrm{Aut}_G(H\backslash G, H\backslash G)$, and it remains only to identify the kernel. But $\gamma \in N_G(H)$ lies in the kernel if $Hg \mapsto H\gamma g$ is the identity map of $H\backslash G$, which happens just if $\gamma \in H$. So we conclude that the kernel is H. ■

The quotient group $N_G(H)/H$ is known as the **Weyl group** of H in G and is sometimes denoted $W_G(H)$.

Fri, Feb. 9

2.3. **The classification of covers.** We have almost shown that working with covers of B is the same as working with transitive right G-sets (technically, we are heading to an "equivalence of categories"). All that is left is to show that for every G-orbit F, there is a cover $p : E \longrightarrow B$ whose fiber is F as a G-set.

We assume that B has a universal cover $q : X \longrightarrow B$. Recall that we showed in Theorem 1.25 that the group of deck transformations of X is isomorphic to G.

Proposition 2.19. *The (left) action of G on X via deck transformations is free and properly discontinuous.*

Proof. Let $x \in X$ and suppose $gx = x$ for some $g \in G$. Recall that here g is a covering homomorphism $X \longrightarrow X$ and thus a lift of $q : X \longrightarrow B$. By the uniqueness of lifts, since g looks like the identity at the point x, it must be the identity. This shows the action is free.

Again, let $x \in X$. We want to find a neighborhood V of x such that only finitely many translates gV meet V. Consider $b = q(x)$. Let U be an evenly-covered neighborhood of b. Then $q^{-1}(U) \cong \coprod V_i$, and $x \in Vj$ for some j. Recall that G freely permutes the pancakes V_i. In particular, the only translate of V_j that meets V_j is the identity translate eV_j. ■

According to Homework IV.2, this means that the quotient map $X \longrightarrow G\backslash X$ is a cover. Actually, the cover $X \xrightarrow{q} B$ factors through a homeomorphism $G\backslash X \cong B$. If we consider the action of a subgroup $H \leq G$, it is still free and properly discontinuous. So we get a covering

$$q_H : X \longrightarrow H\backslash X = X_H$$

for every H. Moreover, the universal property of quotients gives an induced map

$$p_H : H\backslash X \longrightarrow B.$$

Proposition 2.20. *The map $p_H : H\backslash X \longrightarrow B$ is a covering map, and the fiber F is isomorphic to $H\backslash G$ as a G-set.*

Proof. Let $b \in B$. Then we have a neighborhood U which is evenly-covered by q. Recall again that the G-action, and therefore also the H-action, simply permutes the pancakes in $p^{-1}(U)$. We thus get an action of H on the indexing set $\mathcal{I}$ for the pancakes in $p^{-1}(U)$. If we write $W_i = q_H(V_i)$, we thus have the diagram

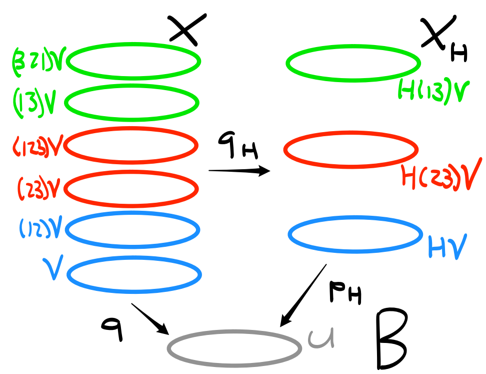
$$\begin{array}{ccccc} q^{-1}(U) & \xrightarrow{q_H} & p_H^{-1}(U) & \xrightarrow{p_H} & U \\ \uparrow \cong & & \uparrow \cong & & \| \\ \coprod_{i \in \mathcal{I}} V_i & \longrightarrow & \coprod_{j \in H\backslash \mathcal{I}} W_j & \longrightarrow & U \end{array}$$

To see that the restriction of p_H to a single W_j gives a homeomorphism, we use the fact that $q_H : V_j \longrightarrow W_j$ is a homeomorphism, since $q_H : X \longrightarrow X_H$ is a covering, and that $q : V_j \longrightarrow U$ is a homeomorphism. It follows that $p_H = q \circ q_H^{-1}$ is a homeomorphism.

For the identification of the fiber $F \subseteq X_H$, notice that the H-action on X acts on each fiber separately, and the quotient of this action on the fiber of X gives precisely $H\backslash G$. ■

Example 2.21. Suppose that $G = \Sigma_3$, the symmetric group on 3 letters, and let $H = \{e, (12)\} \leq G$. If we take an evenly-covered neighborhood U in B, then the situation described in the proof above is given in the picture to the right.

As an aside, note that X_H here is an example of a covering in which the deck transformations do *not* act transitively on the fibers.



To sum up, we have shown that *if* B has a universal cover, then the assignment $(E, p) \mapsto F$ gives an "equivalence of categories" between coverings of B (Cov_B) and G-orbits (Orb_G).