Mon, Feb. 12

We can form a category Cov_B whose objects are the covers of B and whose morphisms are the maps of covers. We can also form a category Orb_G whose objects are the transitive (right) G-sets. Our recent discussion has shown that the assignment (technically 'functor')

$$\mathrm{Cov}_B \longrightarrow \mathrm{Orb}_G, \qquad (E,p) \mapsto F := p^{-1}(b_0)$$

is an equivalence of categories. This means that

(1) (fully faithful) We have a bijection $\mathrm{Cov}_B(E,E') \cong \mathrm{Orb}_G(F,F')$
(2) (essentially surjective) Every G-orbit arises in this way, meaning that any G-orbit is isomorphic to $p^{-1}(b_0)$ for some cover of B.

One consequence of having an equivalence of categories is that this produces a bijection between isomorphism classes of objects.

Corollary 2.22. *The fiber functor* $\mathrm{Cov}_B \longrightarrow \mathrm{Orb}_G$ *induces a bijection*

$$\begin{aligned}\{\textit{isomorphism classes of covers}\} &\cong \{\textit{isomorphism classes of orbits}\}\\ &\cong \{\textit{conjugacy classes of subgroups of } G\ \}\end{aligned}$$

Note that there is no obvious choice of functor in the other direction. Given a G-orbit X, picking a point in the orbit produces an isomorphism to some $H\backslash G$, and then Proposition 2.20 produces a cover whose fiber is isomorphic to X. But this really does involve making a choice. This is a pretty typical situation: a functor that is essentially surjective and fully faithful is called an equivalence of categories, but to produce a functor that looks like an inverse, choices need to be made.

2.4. **Existence of universal covers.** The last result we need to tie this story together is the existence of universal covers.

Definition 2.23. Let B be any space. A subset $U \subseteq B$ is **relatively simply connected** (in B) if every loop in U is contractible in B. We say that B is **semilocally simply connected** if every point has a relatively simply connected neighborhood.

Remark 2.24. Note that if B is very connected and semilocally simply connected, then every point has a path-connected, relatively simply connected neighborhood. This is because if $b\ inU$ is relatively simply connected, then the path component of b in U is open (B is locally path-connected) and also relatively simply-connected (true of any subset of a relatively simply connected subset).

Theorem 2.25. *Let B be very connected. Then there exists a universal cover $X \longrightarrow B$ if and only if B is semilocally simply connected.*

♡♡♡ **Wed, Feb. 14** ♡♡♡

Proof. The forward implication is left as an exercise. For convenience, we fix a basepoint $b_0 \in B$.

We start by working backwards. That is, suppose that $q : X \longrightarrow B$ exists. Given a point $b \in B$, what can we say about the fiber $q^{-1}(b)$? Pick a basepoint $x_0 \in q^{-1}(b_0)$. Then, for each $f \in q^{-1}(b)$, we get a (unique) path-homotopy class of paths $\alpha : x_0 \rightsquigarrow f$. Composing with the covering map q gives a (unique) path-homotopy class of paths $q \circ \alpha : b_0 \rightsquigarrow b$. This now gives a description of the fiber $q^{-1}(b)$ purely in terms of B.

We now take this as a starting point. As a set, we take X to be the set of path-homotopy classes of paths starting at b_0. The map $q : X \longrightarrow B$ takes a class $[\gamma]$ to the endpoint $\gamma(1)$. It remains to (1) topologize X, (2) show that q is a covering map, and (3) show that X is simply-connected.

We specify the topology on X by giving a basis. Let γ be a path in B starting at b_0. Let U be any path-connected, relatively simply-connected neighborhood of the endpoint $\gamma(1)$. Define a subset $U[\gamma] \subseteq X$ to be the set of equivalence classes of paths of the form $[\gamma\delta]$, where $\delta : I \longrightarrow U$ is a path in U. These cover X since each $[\gamma]$ is contained in some $U[\gamma]$ by Remark 2.24. Now suppose that $\gamma \in U_1[\gamma_1] \cap U_2[\gamma_2]$. Then the path-component of $\gamma(1)$ in $U_1 \cap U_2$ is again path-connected and relatively simply connected. Thus

$$\gamma \in U[\gamma] \subseteq U_1[\gamma_1] \cap U_2[\gamma_2].$$

We have shown that the $U[\gamma]$ give a basis for a topology on X.

Next, we show that q is continous. Let $V \subseteq B$ be open and let $q([\gamma]) \in V$, so that $\gamma(1) \in V$. Then we can find a path-connected, relatively simply connected U satisfying $\gamma(1) \in U \subseteq V$. So $U[\gamma]$ is a neighborhood of $[\gamma]$ in $q^{-1}(V)$, as desired.

Since B is path-connected, it follows that q is surjective. Let $b \in B$ and let $b \in U$ be a path-connected, relatively simply-connected neighborhood. We claim that U is evenly covered by q. First, we claim that

$$q^{-1}(U) = \bigcup_{[\gamma] \in q^{-1}(b)} U[\gamma].$$

It is clear that the RHS is contained in the LHS. Suppose that $q([\alpha]) \subseteq U$. Then $\alpha(1) \in U$ and we may pick a path $\delta : \alpha(1) \rightsquigarrow b$ in U. Then $\alpha \in U[\alpha\delta]$.

(This will be continued on Monday ...) ■

Fri, Feb. 16

Exam day!!