Mon, Feb. 19

Proof. (Continued ...) Finally, we wish to show that this is a disjoint union. By the definition of the topology on X, each $U[\gamma]$ is open. Thus suppose that $[\alpha] \in U[\gamma_1] \cap U[\gamma_2]$. This means that

$$[\alpha] = [\gamma_1\delta_1] = [\gamma_2\delta_2].$$

In other words,

$$[\gamma_1\delta_1\overline{\delta_2}] = [\gamma_2].$$

Since U is relatively simply-connected, this implies that $[\gamma_1] = [\gamma_2]$. So any two overlapping $U[\gamma]$ are in fact the same. To finish the proof that q is a covering, we need to show that q restricts to a homeomorphism $q : U[\gamma] \xrightarrow{\cong} U$. Surjectivity follows from the assumption that U is path-connected. Injectivity is the relatively simply-connected hypothesis. Finally, q takes any basis $V[\lambda]$ to the open set V (since V is path-connected), so it is open. We have shown that q is a covering map.

The final step is to show that X is very connected and simply connected. Since X is locally homeomorphic to B and B is locally path-connected, it follows that the same is true of X. Next, we show that X is path-connected (and therefore connected). Let $[\gamma] \in X$. We define a path h in X from the constant path $[c_{b_0}]$ to $[\gamma]$ by $h(s) = [\gamma|_{[0,s]}]$. In the interest of time, we skip the verification that h is continuous (but see Lee, proof of Theorem 11.43).

To see that X is simply connected, let Γ be a loop in X at the basepoint $[c_{b_0}]$. Write $\gamma = q \circ \Gamma$. Then Γ is a lift of γ, but so is the loop $s \mapsto [\gamma_{[0,s]}]$. By uniqueness of lifts, $[\Gamma(s)] = [\gamma_{[0,s]}]$. Then, since Γ is a loop, we have

$$[\gamma] = [\gamma_{[0,1]}] = [\Gamma(1)] = [\Gamma(0)] = [\gamma_{[0,0]}] = [c_{b_0}].$$

In other words, γ is null. Since q is a covering, this implies that Γ is null as well. ∎

We have shown that if a space is **semilocally simply-connected**, then it has a universal cover. So to provide an example of a space without a universal cover, it suffices to give an example of a space with a point which has no relatively simply connected neighborhood.

Example 2.26 (The Hawaiian earring)**.** Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0)$. So each such circle is tangent to the y-axis at the origin. Let $C = \cup_n C_n$. We claim that the origin has no relatively simply connected neighborhood. Indeed, let U be any neighborhood of the origin. Then for large enough n, the circle C_n is contained in U. A loop α that goes once around the circle C_n is not contractible in C. To see this, note that the map $r_n : C \longrightarrow S^1$ which collapses every circle except for C_n is a retraction. The loop $r \circ \alpha$ is not null, so α can't be null.

This example looks like an infinite wedge of circles, but it is not just a wedge. For instance, in each C_n consider an open interval U_n of radian length $1/n$ centered at the origin (or the open left semicircle, if you prefer). The union $U = \cup_n U_n$ of the U_n's is open in the infinite wedge of circles but not in C, since no ϵ-neighborhood of the origin is contained in U.

Wed, Feb. 21

3. The van Kampen Theorem

The focus of the next unit of the course will be on computation of fundamental groups.

One example we have already studied is the fundamental group of $S^1 \vee S^1$. We saw that this is the free group on two generators. We will see similarly that the fundamental group of $S^1 \vee S^1 \vee S^1$ is a free group on three generators. We will also want to compute the fundamental group of the two-holed torus (genus two surface), the Klein bottle, and more.

The main idea will be to decompose a space X into smaller pieces whose fundamental groups are easier to understand. For instance, if $X = U \cup V$ and we understand $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$, we might hope to recover $\pi_1(X)$.

Proposition 3.1. *Suppose that $X = U \cup V$, were U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then the smallest subgroup of $\pi_1(X)$ containing the images of both $\pi_1(U)$ and $\pi_1(V)$ is $\pi_1(X)$ itself.*

In group theory, we would say $\pi_1(X) = \pi_1(U)\pi_1(V)$.

Note that we really do need the assumption that $U \cap V$ is path-connected. If we consider U and V to be open arcs that together cover S^1, then both U and V are simply-connected, but their intersection is not path-connected. Note that here that the product of two trivial subgroups is not $\pi_1(S^1) \cong \mathbb{Z}$!

Proof. Let $\gamma : I \longrightarrow X$ be a loop at x_0. By the Lebesgue number lemma, we can subdivide the interval I into smaller intervals $[s_i, s_{i+1}]$ such that each subinterval is taken by γ into either U or V. We write γ_1 for the restriction of γ to the first subinterval. Suppose, for the sake of argument, that γ_1 is a path in U and that γ_2 is a path in V. Since $U \cap V$ is path-connected, there is a path δ_1 from $\gamma_1(1)$ to x_0. We may do this for each γ_i. Then we have

$$[\gamma] = [\gamma_1] * [\gamma_2] * [\gamma_3] * \cdots * [\gamma_n] = [\gamma_1 * \delta_1] * [\delta_1^{-1} * \gamma_2 * \delta_2] * \cdots * [\delta_{n-1}^{-1} * \gamma_n]$$

This expresses the loop γ as a product of loops in U and loops in V. ■

This is a start, but it is not the most convenient formulation. In particular, if we would like to use this to calculate $\pi_1(X)$, then thinking of the product of $\pi_1(U)$ and $\pi_1(V)$ *inside of* $\pi_1(X)$ is not so helpful. Rather, we would like to express this in terms of some external group defined in terms of $\pi_1(U)$ and $\pi_1(V)$. We have homomorphisms

$$\pi_1(U) \longrightarrow \pi_1(X), \qquad \pi_1(V) \longrightarrow \pi_1(X),$$

and we would like to put these together to produce a map from some sort of product of $\pi_1(U)$ and $\pi_1(V)$ to $\pi_1(X)$. Could this be the direct product $\pi_1(U) \times \pi_1(V)$? No. Elements of $\pi_1(U)$ commute with elements of $\pi_1(V)$ in the product $\pi_1(U) \times \pi_1(V)$, so this would also be true in the image of any homomorphism $\pi_1(U) \times \pi_1(V) \longrightarrow \pi_1(X)$.

What we want instead is a group freely built out of $\pi_1(U)$ and $\pi_1(V)$. The answer is the **free product** $\pi_1(U) * \pi_1(V)$ of $\pi_1(U)$ and $\pi_1(V)$. Its elements are finite length words $g_1 g_2 g_3 g_4 \dots g_n$, where each g_i is in either $\pi_1(U)$ or in $\pi_1(V)$. Really, we use the reduced words, where none of the g_i is allowed to be an identity element and where if $g_i \in \pi_1(U)$ then $g_{i+1} \in \pi_1(V)$.

Example 3.2. We have already seen an example of a free product. The free group F_2 is the free product $\mathbb{Z} * \mathbb{Z}$.

Example 3.3. Similarly, the free group F_3 on three letters is the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Example 3.4. Let C_2 be the cyclic group of order two. Then the free product $C_2 * C_2$ is an infinite group. If we denote the nonidentity elements of the two copies of C_2 by a and b, then elements of $C_2 * C_2$ look like a, ab, $ababa$, $ababababa$, $bababa$, etc.

Note that there is a homomorphism $C_2 * C_2 \longrightarrow C_2$ that sends both a and b to the nontrivial element. The kernel of this map is all words of even length. This is the (infinite) subgroup generated by the word ab (note that $ba = (ab)^{-1}$). In other words, $C_2 * C_2$ is an extension of C_2 by the infinite cyclic group $\mathbb{Z}$. Another way to say this is that $C_2 * C_2$ is a semidirect product of C_2 with $\mathbb{Z}$.

The free product has a universal property, which should remind you of the property of the disjoint union of spaces $X \amalg Y$. First, for any groups H and K, there are inclusion homomorphisms $H \longrightarrow H * K$ and $K \longrightarrow H * K$.

Proposition 3.5. *Suppose that G is any group with homomorphisms $\varphi_H : H \longrightarrow G$ and $\varphi_K : K \longrightarrow G$. Then there is a (unique) homomorphism $\Phi : H * K \longrightarrow G$ which restricts to the given homomorphisms from H and K.*

In other words, the free product is the coproduct in the world of groups.

So Proposition 3.1 can be restated as follows:

Proposition 3.6 (weak van Kampen)**.** *Suppose that $X = U \cup V$, where U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then the natural homomorphism*

$$\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$$

is surjective.

Fri, Feb. 23

Now that we have a surjective homomorphism to $\pi_1(X)$, the next step is to understand the kernel N. Indeed, then the First Isomorphism Theorem will tell us that $\pi_1(X) \cong \big(\pi_1(U) * \pi_1(V)\big)/N$. Here is one way to produce an element of the kernel. Consider a loop α in $U \cap V$. We can then consider its image $\alpha_U \in \pi_1(U)$ and $\alpha_V \in \pi_1(V)$. Certainly these map to the same element of $\pi_1(X)$, so $\alpha_U \alpha_V^{-1}$ is in the kernel.

Proposition 3.7. *With the same assumptions as above, the kernel K of $\pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$ is the normal subgroup N generated by elements of the form $\alpha_U \alpha_V^{-1}$.*

Recall that the normal subgroup generated by the elements $\alpha_U \alpha_V^{-1}$ can be characterized either as (1) the intersection of all normal subgroups containing the $\alpha_U \alpha_V^{-1}$ or (2) the subgroup generated by all conjugates $g\alpha_U \alpha_V^{-1} g^{-1}$.

We will put off the proof of Propostion 3.7 for the moment. Assembling these recent results gives the van Kampen theorem:

Theorem 3.8 (Van Kampen)**.** *Suppose that $X = U \cup V$, where U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then*

$$\pi_1(X, x_0) \cong \big(\pi_1(U, x_0) * \pi_1(V, x_0)\big)/N,$$

where $N \trianglelefteq \pi_1(U, x_0) * \pi_1(V, x_0)$ is the normal subgroup generated by elements of the form $\iota_U(\alpha)\iota_V(\alpha)^{-1}$, for $\alpha \in \pi_1(U \cap V, x_0)$.

There is another, more elegant, way to state the Van Kampen theorem.

Definition 3.9. Suppose given a pair of group homomorphisms $\varphi_G : H \longrightarrow G$ and $\varphi_K : H \longrightarrow K$. We define the **amalgamated free product** (or simply amalgamated product) to be the quotient

$$G *_H K = (G * K)/N,$$

where $N \trianglelefteq G * K$ is the normal subgroup generated by elements of the form $\varphi_G(h)\varphi_K(h)^{-1}$.

It is easy to check that *the amalgamated free product satisfies the universal property of the pushout in the category of groups.*

Theorem 3.10 (Van Kampen, restated)**.** *Let X be given as a union of two open, path-connected subsets U and V with path-connected intersection $U \cap V$. Then the inclusions of U and V into X induce an isomorphism*

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \xrightarrow{\cong} \pi_1(X).$$

Since the pasting lemma tells us that in this situation, X can itself be written as a pushout, the Van Kampen theorem can be interpreted as the statement that, under the given assumptions, the fundamental group construction takes a pushout of spaces to a pushout of groups.

One important special case of this result is when $U \cap V$ is simply connected.

Example 3.11. Take $X = S^1 \vee S^1$. Take U to be an open set containing one of the circles, plus an ϵ-ball around the basepoint in the other circle, and similarly for V with regard to the other circle. Then the intersection $U \cap V$ looks like an 'X' and is contractible, and U and V are both equivalent to S^1. We conclude from this that

$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} \cong F_2.$$

Example 3.12. Take $X = S^1 \vee S^1 \vee S^1$. We can take U to be a neighborhood of $S^1 \vee S^1$ and V to be a neighborhood of the remaining S^1. Then

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong (\mathbb{Z} * \mathbb{Z}) * \mathbb{Z} \cong F_3.$$

Example 3.13. Take $X = S^1 \vee S^2$. Take U to be a neighborhood of S^1 and V to be a neighborhood of S^2. We conclude from this that

$$\pi_1(S^1 \vee S^2) \cong \pi_1(S^1) * \pi_1(S^2) \cong \mathbb{Z}.$$

A natural question now is whether $\pi_1(X \vee Y)$ is always the free product of the $\pi_1(X)$ and $\pi_1(Y)$. Not quite, but a mild assumption allows us to make the conclusion. Note that in the $S^1 \vee S^1$ example, we needed to know that the neighborhoods U and V were homotopy equivalent to S^1 (and that the intersection was contractible).

Definition 3.14. We say that $x_0 \in X$ is a **nondegenerate basepoint** for X if x_0 has a neighborhood U such that x_0 is a deformation retract of U.

Proposition 3.15. *Let x_0 and y_0 be nondegenerate basepoints for X and Y, respectively. Then*

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

Proof. Suppose that x_0 is a deformation retract of the neighborhood $N_X \subseteq X$ and that y_0 is a deformation retract of the neighborhood $N_Y \subseteq Y$. Let $U = X \vee N_Y$ and $V = N_X \vee Y$. Then $U \cap V = N_X \vee N_Y$. The retracting homotopies for N_X and N_Y give $U \simeq X$, $V \simeq Y$, and $U \cap V \simeq *$. The van Kampen theorem then gives the conclusion. ■