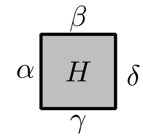
Mon, Feb. 26

Lemma 3.16 (Square Lemma)**.** *Let α, β, γ, and δ be paths in X with*

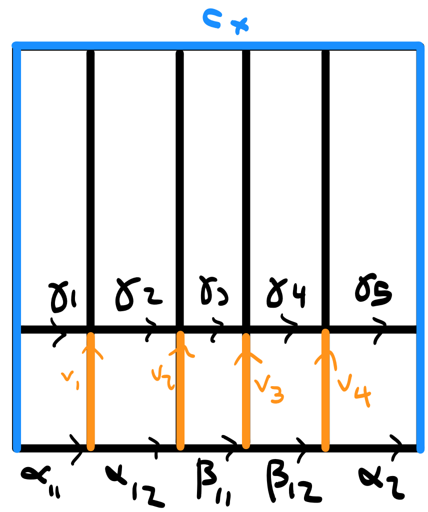
$$\alpha(0) = \gamma(0), \qquad \alpha(1) = \beta(0), \qquad \gamma(1) = \delta(0), \qquad \beta(1) = \delta(1).$$

Then path-homotopies $h : \alpha * \beta \simeq_p \gamma * \delta$ correspond bijectively to maps $H : I^2 \longrightarrow X$ as in the figure.



Proof of Proposition 3.7. Again, it is clear that the kernel K must contain the subgroup N. It remains to show that $K \le N$. Consider an element of K. For simplicity, we assume it is $\alpha_1 \cdot \beta_1 \cdot \alpha_2$, where $\alpha_i \in \pi_1(U)$ and $\beta_1 \in \pi_1(V)$. The assumption that this is in K means that there exists a homotopy $H : I \times I \longrightarrow X$ from the path composition $\alpha_1 * \beta_1 * \alpha_2$ in X to the constant loop.

By the Lebesgue lemma, we may subdivide the square into smaller squares such that each small square is taken by H into either U or V. Again, we suppose for simplicity that this divides α_1 into α_{11} and α_{12} and β_1 into β_{11} and β_{12} (and α_2 is not subdivided).



Note that we cannot write

$$\alpha_1 \cdot \beta_1 \cdot \alpha_2 = \alpha_{11} \cdot \alpha_{12} \cdot \beta_{11} \cdot \beta_{12} \cdot \alpha_2$$

in $\pi_1(U) * \pi_1(V)$ since these are not all loops. But we can fix this, using the same technique as in the proof of Prop 3.1. In other words, we append a path δ back to x_0 at the end of every path on an edge of a square. If that path is in U (or V or $U \cap V$), we take δ in U (or V or $U \cap V$). Also, if the path already begins or ends at x_0, we do not append a δ. For convenience, we keep the same notation, but remember that we have really converted all of these paths to loops.

Let us turn our attention now to the homotopy H on the first (lower-left) square. Either H takes this into U or into V. If it is U, then we get a path homotopy in U $\alpha_{11} \simeq_p \gamma_1 \cdot v_1^{-1}$. If, on the other hand, H takes this into V, then it follows that α_{11} is really in $U \cap V$. This gives us a path homotopy in V $\alpha_{11} \simeq_p \gamma_1 \cdot v_1^{-1}$. But the group element α_{11} comes from $\pi_1(U)$ in the free product $\pi_1(U) * \pi_1(V)$. We would like to replace this with the element α_{11} from $\pi_1(V)$.

Lemma 3.17. *Let γ be any loop in $U \cap V$. Then, in the quotient group $Q = \big(\pi_1(U) * \pi_1(V)\big)/N$, the elements γ_U and γ_V are equivalent.*

Proof. The point is that

$$\gamma_V N = \gamma_U \gamma_U^{-1} \gamma_V N = \gamma_U \cdot \Big((\gamma^{-1})_U (\gamma^{-1})_V^{-1} \Big) N = \gamma_U N.$$

■

From here on out, we work in the quotient group Q. The goal is to show that the original element $\alpha_1 \cdot \beta_1 \cdot \alpha_2$ is trivial in Q. According to the above, we can replace $(\alpha_1)_U(\beta_1)_V(\alpha_2)_U$ with either

$$(\gamma_1)_U (v_1^{-1})_U (\alpha_{12})_U (\beta_{11})_V (\beta_{12})_V (\alpha_2)_U$$

or

$$(\gamma_1)_V (v_1^{-1})_V (\alpha_{12})_U (\beta_{11})_V (\beta_{12})_V (\alpha_2)_U.$$

We then do the same with each of $\alpha_{12}, \ldots, \alpha_2$. The resulting expression will have adjacent terms v_i and v_i^{-1}. For the same i, these two loops may have the same label (U or V) or different labels. But by the lemma, we can always change the label if the loop lies in the intersection. So we get the path-composition of the paths along the top edges of the bottom squares. We then repeat the procedure, moving up rows until we get to the very top. But of course the top edges of the top squares are all constant loops. It follows that we end up with the trivial element (of Q). So $K = N$. ■

The next application is the computation of the fundamental group of any *graph*. We start by specifying what we mean by a graph. Recall that $S^0 \subseteq \mathbb{R}$ is usually defined to be the set $S^0 = \{-1, 1\}$. For the moment, we take it to mean instead $S^0 = \{0, 1\}$ for convenience.

Definition 3.18. A **graph** is a 1-dimensional CW complex.

Of special importance will be the following type of graph.

Definition 3.19. A **tree** is a connected graph such that it is not possible to start at a vertex v_0, travel along successive edges, and arrive back at v_0 without using the same edge twice.

(Give examples and nonexamples)

Proposition 3.20. *Any tree is contractible. Even better, if v_0 is a vertex of the tree T, then v_0 is a deformation retract of T.*

Proof. We give the proof in the case of a finite tree. Use induction on the number of edges. If T has one edge, it is homeomorphic to I. Assume then that any tree with n edges deformation retracts onto any vertex and let T be a tree with $n+1$ edges. Let $v_0 \in T$. Now let $v_1 \in T$ be a vertex that is maximally far away from v_0 in terms of number of edges traversed. Then v_1 is the endpoint of a unique edge e, which we can deformation retract onto its other endpoint. The result is then a tree with n edges, which deformation retracts onto v_0. ■

Wed, Feb. 28

Corollary 3.21. *Any tree is simply connected.*

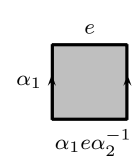
Definition 3.22. If X is a graph and $T \subseteq X$ is a tree, we say that T is a **maximal tree** if it is not contained in any other (larger) tree.

By Zorn's Lemma, any tree is contained in some maximal tree.

Theorem 3.23. *Let X be a connected graph and let $T \subseteq X$ be a maximal tree. The quotient space X/T is a wedge of circles, one for each edge not in the tree. The quotient map $q : X \longrightarrow X/T$ is a homotopy equivalence.*

Proof. Since T contains every vertex, all edges in the quotient become loops, or circles. To see that q is a homotopy equivalence, we first define a map $b : X/T \cong \bigvee S^1 \longrightarrow X$. Recall that to define a continuous map out of a wedge, it suffices to specify the map out of each wedge summand. Fix a vertex $v_0 \in T \subseteq X$. Pick a deformation retraction T down to v_0 as in Proposition 3.20. Then, for each vertex v, the homotopy provides a path $\alpha_v : v_0 \rightsquigarrow v$. Now suppose we have a circle corresponding to the edge e in X from v_1 to v_2. We then send our circle to the loop $\alpha_{v_1} e \alpha_{v_2}^{-1}$.

The composition $q \circ b$ on a wedge summand S^1 looks like $c * \mathrm{id} * c$ and is therefore homotopic to the identity. For the other composition, we want to extend the given homotopy on T to a homotopy on X. For simplicity, we give the argument in the case that $X = T \cup e$ has a single edge not in a maximal tree. We wish to define a homotopy $h : X \times I \longrightarrow X$, but we already have the homotopy on the subspace $T \times I$. It remains to specify the homotopy on $e \times I$, where we already have the homotopy on the edges $e_0 \times I$ and $e_1 \times I$. At time 0, the map $b \circ q$ takes e to $\alpha_1 * e * \alpha_2^{-1}$, whereas at time 1, the identity map takes e to e. We are now done by the Square Lemma (3.16).



■

Corollary 3.24. *The fundamental group of any graph is a free group.*

We will use this to deduce an algebraic result about free groups. But first, a result about coverings of graphs.

Theorem 3.25. *Let $p : E \longrightarrow B$ be a covering, where B is a connected graph. Then E is also a connected graph.*

Proof. Recall our definition of a graph. It is a space obtained by glueing a set of edges to a set of vertices. Let B_0 be the vertices of B and B_1 be the set of edges. Let $E_0 \subseteq E$ be $p^{-1}(B_0)$ and define

$$E_1 \subseteq B_1 \times E_0$$

to be the set of pairs $(\alpha : \{0,1\} \longrightarrow B_0, v)$ such that $\alpha(0) = p(v)$. We then have compatible maps $E_0 \hookrightarrow E$ and $\amalg_{E_1} I \longrightarrow E$. The second map is given by the unique path-lifting property. These assemble to give a continuous map from the pushout

$$f : \hat{E} = E_0 \cup_{\coprod \partial I} \coprod_{e_1} I \longrightarrow E.$$

This pushout is a 1-dimensional CW complex, which is our definition of a graph.

$\boxed{\textbf{Fri, Mar. 2}}$

To see that f is surjective, let $x \in E$. Then $p(x)$ lies in some 1-cell β of B. Pick a path γ in B, lying entirely in β, from $p(x)$ to a vertex b_0. Then γ lifts uniquely to a path $\tilde{\gamma}$ starting at x in E. Write $v = \tilde{\gamma}(1)$. Then x lives in the 1-cell (β, v), so f is surjective.

The restriction of f to E_0 is injective, since this is the inclusion of the subset $E_0 \hookrightarrow E$. If y and z are two points of $\hat{E}$, where z is not a zero-cell and $f(y) = f(z)$, then $pf(y) = pf(z)$ in B. Since $pf(z)$ is not a 0-cell of B, we conclude that y is also not a 0-cell in $\hat{E}$. Now $pf(y)$ amd $pf(z)$ live in the same 1-cell of B, and since $f(y) = f(z)$ in E, uniqueness of lifts tells us that y and z live in the same 1-cell of $\hat{E}$. But the restriction of pf to the interior of this 1-cell is a homeomorphism onto the 1-cell of B. Since $pf(y) = pf(z)$, we conclude that $y = z$.

There are now several arguments for why this must be a homeomorphism. If B is a finite graph and E is a finite covering, we are done since E' is compact and E is Hausdorff (since B is Hausdorff). More generally, the map $\hat{E} \longrightarrow E$ is a map of covers which induces a bijection on fibers, so it must be an isomorphism of covers. ∎

Now here is a purely algebraic statement, which we can prove by covering theory.

Theorem 3.26. *Any subgroup H of a free group G is free. If G is free on n generators and the index of H in G is k, then H is free on $1 - k + nk$ generators.*

Proof. Define B to be a wedge of circles, one circle for each generator of G. Then $\pi_1(B) \cong G$. Let $H \leq G$ and let $p : E \longrightarrow B$ be a covering such that $p_*(\pi_1(E)) = H$. By the previous result, E is a graph and so $\pi_1(E)$ is a free group by the result from last time.

For the second statement, we introduce the **Euler characteristic** of a graph, which is defined as $\chi(B) = \#$ vertices $-\#$ edges. In this case, we have $\chi(B) = 1 - n$. Since H has index k in G, this means that G/H has cardinality k. But this is the fiber of $p : E \longrightarrow B$. So E has k vertices, and each edge of B lifts to k edges in E. So $\chi(E) = k - kn$.

On the other hand, we know from last time that E is homotopy equivalent to E/T, where $T \subseteq E$ is a maximal tree. Note that collapsing any edge in a tree does not change the Euler characteristic. The number of generators, say m of H, is then the number of edges in E/T, so we find that $\chi(E) = 1 - m$. Setting these equal gives

$$k - kn = 1 - m, \qquad \text{or} \qquad m = 1 - k + kn.$$

∎