Mon, Mar. 5

We encountered an important idea in this proof, which can be defined more generally.

Definition 3.27. Let X be a CW complex having finitely many cells in each dimension (we saw that X is finite type). Then the **Euler characteristic** of X is

$$\chi(X) := \sum_{n\geq 0}(-1)^n\#\{n\text{-cells of } X\}.$$

In fact, the number $\chi(X)$ does not depend on the choice of CW structure on X, though this is far from obvious from the definition. We will see Euler characteristics again later in the course.

3.1. **The effect of attaching cells.** The van Kampen Theorem also gives an effective means of computing the fundamental group of a CW complex.

Given a space X and a map $\alpha : S^1 \longrightarrow X$, we may attach a disc along the map α to form a new space

$$X' = X \cup_\alpha D^2.$$

Since the inclusion of the boundary $S^1 \hookrightarrow D^2$ is null, it follows that the composition

$$\alpha : S^1 \longrightarrow X \longrightarrow X'$$

is also null. So we have effectively killed off the class $[\alpha] \in \pi_1(X)$.

We can use the van Kampen theorem to show that this is all that we have done.

Proposition 3.28. *Let X be path-connected and let $\alpha : S^1 \longrightarrow X$ be a loop in X, based at x_0. Write $X' = X \cup_\alpha D^2$. Then*

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X)/[\alpha].$$

Of course, we really mean the normal subgroup generated by α.

Proof. Consider the open subsets U and V of D^2, where $U = D^2 - \overline{B_{1/3}}$ and $V = B_{2/3}$. The map $\iota_{D^2} : D^2 \longrightarrow X'$ restricts to a homeomorphism (with open image) on the interior of D^2, so the image of V in X' is open and path-connected. Now let $U' = X \cup U$. Since this is the image under the quotient map $X \amalg D^2 \longrightarrow X'$ of the saturated open set $X \amalg U$, U' is open in X'. It is easy to see that U' is also path-connected.

Now U' and V cover X'. Since U deformation retracts onto the boundary, it follows that U' deformation retracts onto X. The open set V is contractible. Finally, the path-connected subset $U' \cap V$ deformation retracts onto the circle of radius $1/2$. Moreover, the map

$$\mathbb{Z} \cong \pi_1(U' \cap V) \longrightarrow \pi_1(U') \cong \pi_1(X)$$

sends the generator to $[\alpha]$. The van Kampen theorem then implies that

$$\pi_1(X') \cong \pi_1(X)/\langle\alpha\rangle.$$

■

Actually, we cheated a little bit in this proof, since in order to apply the van Kampen theorem, we needed to work with a basepoint in $U' \cap V$. A more careful proof would include the necessary change-of-basepoint discussion.

What about attaching higher-dimensional cells?

Proposition 3.29. *Let X be path-connected and let $\alpha : S^{n-1} \longrightarrow X$ be an attaching map for an n-cell in X, based at x_0. Write $X' = X \cup_\alpha D^n$. Then, if $n \geq 3$,*

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X).$$

Proof. The proof strategy is the same as for a 2-cell, so we don't reproduce it. The only change is that now $U' \cap V \simeq S^{n-1}$ is simply-connected. ■

Wed, Mar. 7

Example 3.30. If we attach a 2-cell to S^1 along the identity map $\mathrm{id} : S^1 \longrightarrow S^1$, we obtain D^2. We have killed all of the fundamental group. If we attach another 2-cell, we get S^2. Then $\chi(S^2) = 2 - 2 + 2 = 2$.

Attaching a 3-cell to S^2 via $\mathrm{id} : S^2 \longrightarrow S^2$ gives D^3. Attaching a second 3-cell gives S^3. The previous results tells us that all spaces obtained in this way (D^n and S^n) will be simply connected. Here we get $\chi(S^3) = 2 - 2 + 2 - 2 = 0$. More generally, we get

$$\chi(S^n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Example 3.31. ($\mathbb{RP}^n$) A more interesting example is attaching a 2-cell to S^1 along the double covering $\gamma_2 : S^1 \longrightarrow S^1$. Since this loop in S^1 corresponds to the element 2 in $\pi_1(S^1) \cong \mathbb{Z}$, the resulting space X' has $\pi_1(X') \cong \mathbb{Z}/2$. We have previously seen (last semester) that this is just the space $\mathbb{RP}^2$, since $\mathbb{RP}^2$ can be realized as the quotient of D^2 by the relation $x \sim -x$ on the boundary. This presents $\mathbb{RP}^2$ as a cell complex with a single 0-cell (vertex), a single 1-cell, and a single 2-cell. Then $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$.

We can next attach a 3-cell to $\mathbb{RP}^2$ along the double cover $S^2 \longrightarrow \mathbb{RP}^2$. The result is homeomorphic to $\mathbb{RP}^3$ by an analogous argument. By the above, this does not change the fundamental group, so that $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$, and we count $\chi(\mathbb{RP}^3) = 1 - 1 + 1 - 1 = 0$. In general, we have $\mathbb{RP}^n$ given as a cell complex with a single cell in each dimension. We have $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2$ for all $n \geq 2$, and

$$\chi(\mathbb{RP}^n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Example 3.32. ($\mathbb{CP}^n$) Recall that $\mathbb{CP}^1 \cong S^2$ is simply connected. Last semester, we showed that $\mathbb{CP}^n$ has a CW structure with a single cell in every even dimension. For example, $\mathbb{CP}^2$ is obtained from $\mathbb{CP}^1$ by attaching a 4-cell. It follows that every $\mathbb{CP}^n$ is simply-connected, and $\chi(\mathbb{CP}^n) = n+1$.

Let's look at a few more examples of CW complexes.

Example 3.33. (Torus) Attach a 2-cell to $S^1 \vee S^1$ along the map $S^1 \longrightarrow S^1 \vee S^1$ given by $aba^{-1}b^{-1}$, where a and b are the standard inclusions $S^1 \hookrightarrow S^1 \vee S^1$. We saw last semester that the resulting pushout is the torus, presented as a quotient of $D^2 \cong I^2$.

We claim that

$$\pi_1(T^2) \cong F_2/aba^{-1}b^{-1} \cong \mathbb{Z}^2.$$

Proposition 3.34. *The natural map $\varphi : F(a,b) \longrightarrow \mathbb{Z}^2$ defined by $\varphi(a) = (1,0)$ and $\varphi(b) = (0,1)$ induces an isomorphism*

$$F(a,b)/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2.$$

Proof. Let $K = \ker(\varphi)$ and let $N \trianglelefteq F(a,b)$ be the normal subgroup generated by $aba^{-1}b^{-1}$. By the First Isomorphism Theorem, $F(a,b)/K \cong \mathbb{Z}^2$, so it suffices to show that $N = K$. Since $aba^{-1}b^{-1} \in K$, it follows that $N \leq K$. Since $N \trianglelefteq K$, we wish to show that the quotient group K/N is trivial. Let $g = \overline{a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3}} \in K/N$. In K/N, we have $\overline{ab} = \overline{ba}$, so

$$\overline{a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3}} = \overline{a^{n_1+n_2+n_3}b^{k_1+k_2}}.$$

Since $g \in K$, we have $n_1 + n_2 + n_3 = 0$ and $k_1 + k_2 = 0$, so $\overline{g} = e$ in K/N. ■

So the answer coming from the van Kampen theorem matches our previous computation of $\pi_1(T^2)$.

In this cell structure on the torus, there is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell, so that

$$\chi(T^2) = 1 - 2 + 1 = 0.$$

Fri, Mar. 9

Example 3.35. (Klein bottle) One definition of the Klein bottle K is as the quotient of I^2 in which one opposite pair of edges is identified with a flip, while the other pair is identified without a flip. This leads to the computation

$$\pi_1(K) \cong F(a, b)/\langle aba^{-1}b \rangle.$$

For certain purposes, this is not the most convenient description. Cut the square along a diagonal and repaste the triangles along the previously flip-identified edges. The resulting square leads to the computation

$$\pi_1(K) \cong F(a, c)/\langle a^2c^2 \rangle.$$

The equation $c = a^{-1}b$ allows you to go back and forth between these two descriptions.

Like the torus, the resulting cell complex has a single 0-cell, two 1-cells, and a single 2-cell, so

$$\chi(K) = 1 - 2 + 1 = 0.$$

The next example is <u>not</u> obtained by attaching a cell to $S^1 \vee S^1$.

Example 3.36. If we glue the boundary of I^2 according to the relation $abab$, the resulting space can be identified with $\mathbb{RP}^2$. Notice in this case that the four vertices do not all become identified. Rather they are identified in pairs, and we are left with two vertices after making the quotient. This example can be visualized by thinking of identifying the two halves of ∂D^2 via a twist. Using this cell structure, we get

$$\chi(\mathbb{RP}^2) = 2 - 2 + 1 = 1.$$

3.2. **The classification of surfaces.** These 2-dimensional cell complexes are all examples of **surfaces** (compact, connected 2-dimensional manifolds).

There is an important construction for surfaces called the **connected sum**.

Definition 3.37. Suppose M and N are surfaces. Pick subsets $D_M \subseteq M$ and $D_N \subseteq N$ that are homeomorphic to D^2 and remove their interiors from M and N. Write $M' = M - \text{Int}(D_M)$ and $N' = N - \text{Int}(D_N)$. Then the connected sum of M and N is defined to be

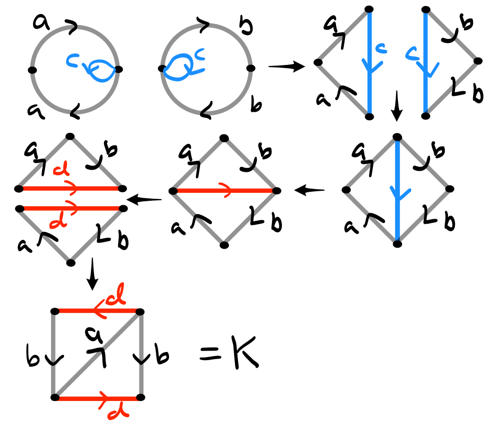
$$M\#N = M' \cup_{S^1} N',$$

where the maps $S^1 \longrightarrow M'$ and $S^1 \longrightarrow N'$ are the inclusions of the boundaries of the removed discs.

Example 3.38. If M is a surface, then the connect sum $M\#S^2$ is again homeomorphic to M.

Proposition 3.39. *The connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle, K.*

Proof. See the figure to the right. ■



Example 3.40. If M is a surface, then the connect sum $M\#T^2$ can be viewed as M with a "handle" glued on.

For example, consider $M = T^2$. Then $T^2\#T^2$ looks liked a "two-holed torus". This is called M_2, the (orientable) surface of genus two. From the cell structure resulting from the picture, we see a wedge of four circles (let's call the generators of the circles a_1, b_1, a_2, b_2) with a two-cell attached along the element $[a_1, b_1][a_2, b_2]$. It follows that the fundamental group of M_2 is

$$F(a_1, b_1, a_2, b_2)/[a_1, b_1][a_2, b_2].$$

We also find that $\chi(M_2) = 1 - 4 + 1 = -2$.