CLASS NOTES
MATH 651 (SPRING 2018)

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Contents

Wed, Jan. 10

Here are a list of main topics for this semester:

(1) the fundamental group (topology $\rightsquigarrow$ algebra) (Hatcher - Ch. 1.1; Lee - Ch. 7, Ch. 8)
(2) the theory of covering spaces (Hatcher - Ch. 1.3; Lee - Ch. 11, Ch. 12)

Example 0.1.

(a) What spaces cover $\mathbb{R}$? Only $\mathbb{R}$ itself. Every covering map $E \longrightarrow \mathbb{R}$ is a homeomorphism.
(b) What spaces cover S^1? There is the n-sheeted cover of S^1 by itself, for any nonzero integer n. (Wrap the circle around itself n times.) There is also the exponential map $\mathbb{R} \longrightarrow S^1$.
(c) What spaces cover S^2? Only S^2 itself. Every covering map $E \longrightarrow S^2$ is a homeomorphism.

Date: May 1, 2018.

(d) What spaces cover $\mathbb{RP}^2$? There is the defining quotient map $S^2 \longrightarrow \mathbb{RP}^2$ and the homeomorphisms.

(3) computation of the fundamental group via the Seifert-van Kampen theorem. (Hatcher - Ch. 1.2, Lee - Ch. 9, Ch. 10)
(4) classification of surfaces (compact, connected) and the Euler characteristic. (Lee - Ch. 6, Ch. 10)
(5) homology of CW complexes (Hatcher - Ch. 2.1, Lee - Ch. 13)

The fundamental group, an algebraic object, will turn out to be crucial for understanding topics in geometric topology (coverings, surfaces).

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1. The fundamental group - Examples

Our first major result in the course will be the computation of the fundamental group of the circle. In particular, we will show that it is nontrivial! The argument will involve a number of new ideas, and one thing I hope you will learn from this course is that **computing fundamental groups is hard**!

1.1. The fundamental group of S^1.

Today, we begin the discussion of the fundamental group of S^1. We will need the following technical result that could have been included in the fall semester.

Proposition 1.1. *(Lebesgue number lemma)[Lee, 7.18] Let $\mathcal{U}$ be an open cover of a compact metric space X. Then there is a number $\delta > 0$ such that any subset $A \subseteq X$ of diameter less than δ is contained in an open set from the cover.*

For any n, consider the loop in S^1 given by $\gamma_n(t) = e^{2\pi i n t}$. For today, we will denote the standard basepoint of S^1, the point $(1,0)$, by the symbol $\star$.

Theorem 1.2. *The assignment $n \mapsto \gamma_n$ is an isomorphism of groups*

$$\Gamma : \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, \star).$$

Proof. Let's start by showing that it is a homomorphism. First note that γ_0 is the constant path at 1, which is the identity element of the fundamental group. Also, note that γ_{-n} is the path-inverse of γ_n. It then remains to show that the path $\gamma_n \cdot \gamma_k$ is path-homotopic to γ_{n+k} when n and k are non-negative.

For any $0 \leq c \leq 1$, we can define a path which first traverses γ_n on the time interval $[0, c]$ and then traverses γ_k on the time interval $[c, 1]$. Any two choices of c gives homotopic paths. The choice $c = 1/2$ gives the usual path-composition $\gamma_n \cdot \gamma_k$, whereas the choice $c = n/(n+k)$ gives γ_{n+k}.

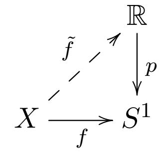
To show that Γ is also a bijection, we will rely on the exponential map

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ t &\mapsto e^{2\pi i t}. \end{aligned}$$

Note that $p^{-1}(\star) = \mathbb{Z}$. One important property of this map that we will need is that we can cover S^1, say using the open sets $U_1 = S^1 \setminus \{(1,0)\}$ and $U_2 = S^1 \setminus \{(-1,0)\}$. On each of these open sets

U_i, the preimage $p^{-1}(U_i)$ is a (countably infinite) disjoint union of subsets $V_{i,j}$ of $\mathbb{R}$, and p restricts to a homeomorphism $p : V_{i,j} \cong U_i$.

If $f : X \longrightarrow S^1$ is a map from some space X, then by a **lift** $\tilde{f} : X \longrightarrow \mathbb{R}$ we mean simply a map such that $p \circ \tilde{f} = f$.

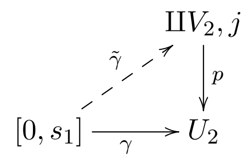


Fri, Jan. 12

Lemma 1.3. *Let $\gamma : I \longrightarrow S^1$ be a loop at $\star$ and let $n \in \mathbb{Z}$. Then there is a unique lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ such that $\tilde{\gamma}(0) = n$.*

Proof. By the Lebesgue number lemma applied to I, there is a subdivision of I into subintervals $[s_i, s_{i+1}]$ such that each subinterval is contained in a single $\gamma^{-1}(U_i)$.

Consider the first such subinterval $[0, s_1] \subseteq \gamma^{-1}(U_2)$. Now our lifting problem simplifies to that on the right. The interval $[0, s_1]$ is connected, so the image of $\tilde{\gamma}$ must lie in a single component $V_{1,j}$. And we have no choice of the component since we have already decided that $\tilde{\gamma}(0)$ must be n. Call the component $V_{2,0}$.



Now our lifting problem reduces to lifting against the *homeomorphism* $p_{2,0} : V_{2,0} \cong U_2$, and we define our lift on $[0, s_1]$ to be the composite $p_{2,0}^{-1} \circ \gamma$. Now play the same game with the next interval $[s_1, s_2]$. We already have a lift at the point s_1, so this forces the choice of component at this stage. By induction, at each stage we have a unique choice of lift on the subinterval $[s_k, s_{k+1}]$. Piecing these all together gives the desired lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$. ■

Thus given a loop γ at $\star$, there is a unique lift $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ that starts at 0. The endpoint of the lift $\tilde{\gamma}$ must also be in $p^{-1}(0) = \mathbb{Z}$. We claim that the function $\gamma \mapsto w(\gamma) = \tilde{\gamma}(1)$ is inverse to Γ. First we must show it is well-defined.

Lemma 1.4. *Let $h : \gamma \simeq_p \delta$ be a path-homotopy between loops at $\star$ in S^1. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow \mathbb{R}$ such that $\tilde{h}(0, 0) = 0$.*

Proof. We already know about the unique lift $\tilde{\gamma}$ on $I \times 0$. On $0 \times I$, the only possible lift is the constant lift. Now use the Lebesgue number lemma again to subdivide the compact square $I \times I$ so that every subsquare is mapped by γ into one of the U_i. Using the same argument as above, we get a unique lift on each subsquare, starting from the bottom left square and moving along each row systematically. ■

Note that the lift $\tilde{h}$ is a path-homotopy between the lifts $\tilde{\gamma}$ and $\tilde{\delta}$. This is because $\tilde{h}(0, t)$ and $\tilde{h}(1, t)$ are lifts of constant paths. By the uniqueness of lifts, according to Lemma 1.3, the lift of a constant path must be a constant path. It follows that $\tilde{\gamma}(1) = \tilde{\delta}(1)$. This shows that the function $w : \pi_1(S^1) \longrightarrow \mathbb{Z}$ is well-defined.

It remains to show that w is the inverse of Γ.

First note that $\delta_n(s) = ns$ is a path in $\mathbb{R}$ starting at 0, and $p \circ \delta_n(s) = e^{2\pi i(ns)} = \gamma_n(s)$, so δ_n is a lift of γ_n starting at 0. By uniqueness of lifts (Lemma 1.3), δ_n must be $\tilde{\gamma_n}$. Therefore

$$w \circ \Gamma(n) = w(\gamma_n) = \tilde{\gamma}_n(1) = \delta(1) = n.$$

It remains to check that $\left[\Gamma(w(\gamma))\right] = [\gamma]$ for any loop γ. Consider lifts $\widetilde{\Gamma(w(\gamma))}$ and $\tilde{\gamma}$. These are both paths in $\mathbb{R}$ starting at 0 and ending at $\tilde{\gamma}(1) = w(\gamma)$ (this uses that $w \circ \Gamma(n) = n$). But any two such paths are homotopic (use a straight-line homotopy)! Composing that homotopy with the exponential map p will produce a path-homotopy $\Gamma(w(\gamma)) \simeq_p \gamma$ as desired. ■

Wed, Jan. 17

Using problem 4 from Homework I, we get the following result.

Corollary 1.5. *Let T^n denote the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times). Then $\pi_1(T^n) \cong \mathbb{Z}^n$.*

Theorem 1.6. *(Borsuk-Ulam Theorem) For every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, there is an antipodal pair of points $\{x, -x\} \subset S^2$ such that the $f(x) = f(-x)$.*

Proof. Suppose not. Then we can define a map $g : S^2 \longrightarrow S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

Then g satisfies $g(-x) = -g(x)$. Let $\gamma : S^1 \longrightarrow S^1$ be the restriction to the equator. Note that since γ extends over the northern (or southern) hemisphere, the loop γ is null. We also write δ for the composition $I \longrightarrow S^1 \xrightarrow{\gamma} S^1$.

The equation $g(-z) = -g(z)$ means that $\gamma(-z) = -\gamma(z)$ or $\delta(t + \frac{1}{2}) = -\delta(t)$. Denote by $\tilde{\delta}$ a lift to a path in $\mathbb{R}$. Then $\tilde{\delta}$ must satisfy the equation $\tilde{\delta}(t + \frac{1}{2}) = \tilde{\delta}(t) + \frac{1}{2} + k$ for some integer k. In particular, we find that

$$\tilde{\delta}(1) = \tilde{\delta}\left(\frac{1}{2}\right) + \frac{1}{2} + k = \tilde{\delta}(0) + 1 + 2k.$$

Thus the degree of γ is the odd integer $1 + 2k$. This contradicts that γ is null. ∎

Application: At any point in time, there are two polar opposite points on Earth having the same temperature and same barometric pressure. (Or pick any two continuously varying parameters)

Corollary 1.7. *The sphere S^2 is not homeomorphic to any subspace of $\mathbb{R}^2$.*

Proof. According to the theorem, there is no continuous injection $S^2 \longrightarrow \mathbb{R}^2$. ∎

1.2. **Fundamental group of spheres.** We saw that S^1 has a nontrivial fundamental group, but in contrast we will see that the higher spheres all have trivial fundamental groups. A (path-connected) space with trivial fundamental group is said to be **simply connected**.

Theorem 1.8. *The n-sphere S^n is simply connected if $n \geq 2$.*

This follows from the following theorem.

Theorem 1.9. *Any continuous map $S^1 \longrightarrow S^n$ is path-homotopic to one that is not surjective.*

Let's first use this to deduce the statement about n-spheres. Let γ be a loop in S^n. We know it is path-homotopic to a loop δ that is not surjective. But recall that $S^n - \{P\} \cong \mathbb{R}^n$. Thus we can contract δ using a straight-line homotopy in the complement of any missed point. It remains to prove the latter theorem.

Proof. There are a number of ways to prove this result. For instance, it is an easy consequence of "Sard's Theorem" from differential topology. Here is a proof using once again the Lebesgue number lemma.

Let $\{U, V\}$ be the covering of S^n, where U is the upper (open) hemisphere, and V is the complement of the North pole. Let $\gamma : S^1 \longrightarrow S^n$ be a loop. By Lebesgue, we can subdivide the interval I into finitely many subintervals $[s_i, s_{i+1}]$ such that on each subinterval, γ stays within either U or V. We will deform γ so that it misses the North pole. On the subintervals that are mapped into V, nothing needs to be done.

Suppose $[s_i, s_{i+1}]$ is not mapped into V, so that γ passes through the North pole on this segment. Recall that the open hemisphere U is homeomorphic to $\mathbb{R}^n$. The problem thus reduces to the following: given a path in $\mathbb{R}^n$, show it is path-homotopic to one not passing through the origin.

This is simple. First, any path is homotopic to the straight-line path. If that does not pass through the origin, great. If it does, just wiggle it a little, and it won't any more. ■

Corollary 1.10. *The infinite sphere S^∞ is simply connected.*

Proof. Consider a loop α in S^∞. The image of α is then a compact subset of the CW complex S^∞. It follows (see Hatcher, A.1) that the image of α is contained in a finite union of cells. In other words, the image of α is contained in some S^n. By the above, α is null-homotopic in S^n and therefore in S^∞ as well. ■

Fri, Jan. 19

You showed on your homework that S^∞ is contractible, and this in fact implies simply connected, as the next result shows.

Theorem 1.11. *Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then, for any choice of basepoint $x \in X$, the induced map*

$$f_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(Y, f(x))$$

is an isomorphism.

At first glance, this might seem obvious, since we have a quasi-inverse $g : Y \longrightarrow X$ to f, and so we would expect g_* to be the inverse of f_*. But note that there is no reason that $g(f(x))$ would be x again, so g_* does not even map to the correct group to be the inverse of f_*. We need to employ some sort of change-of-basepoint to deal with this. So we take a little detour to address this issue.

1.3. **Dependence on the basepoint.**

Although we often talk about "the fundamental group" of a space X, this group depends on the choice of basepoint for the loops. One thing at least should be clear: if we want to understand $\pi_1(X, x_0)$, only the path component of x_0 in X is relevant. Any other path component can be ignored. More precisely, if PC_x denotes the path-component of a point x, then for any choice of basepoint x_0, we get an **isomorphism of groups**

$$\pi_1(PC_{x_0}, x_0) \cong \pi_1(X, x_0).$$

For this reason, we will often assume from now on that our spaces are path-connected.

Under this assumption that X is path-connected, how does the fundamental group depend on the choice of base point? Suppose that x_0 and x_1 are points in X. How can we compare loops based at x_0 to loops based at x_1? Since X is path-connected, we may choose some path α in X from x_0 to x_1. Then we may use the change-of-basepoint technique that we discussed at the end of the fall semester. If γ is a loop based at x_0, we get a loop $\overline{\alpha} \cdot \gamma \cdot \alpha$ based at x_1. Let us write $\Phi_\alpha(\gamma)$ for this loop. The same argument we gave in the case $X = S^1$ generalizes to give

Proposition 1.12.

1. *The operation Φ_α gives a well-defined operation on homotopy-classes of loops.*
2. *The operation Φ_α only depends on the homotopy-class of α.*
3. *The operation Φ_α induces an isomorphism of groups*

$$\Phi_\alpha : \pi_1(X, x_0) \cong \pi_1(X, x_1)$$

with inverse induced by $\Phi_{\overline{\alpha}}$.

So, as long as X is path-connected, the isomorphism-type of the fundamental group of X does not depend on the basepoint. For example, once we know that $\pi_1(\mathbb{R}^2, \mathbf{0}) = \langle e \rangle$, it follows that the same would be true with any other choice of basepoint. More generally, we know that any convex subset of $\mathbb{R}^n$ is simply connected.

Proposition 1.13. *Let h be a homotopy between maps $f, g : X \rightrightarrows Y$. For a chosen basepoint $x_0 \in X$, define a path α in Y by $\alpha(s) = h(x_0, s)$. Then the diagram to the right commutes.*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\ & \searrow^{g_*} & \cong \downarrow \Phi_\alpha \\ & & \pi_1(Y, g(x_0)) \end{array}$$

Proof. For any loop γ in X based at x_0, we want a path-homotopy $H : \Phi_\alpha(f \circ \gamma) \simeq_p g \circ \gamma$. For convenience, let us write $y_0 = g(x_0)$. For each t, let α_t denote the path $\alpha_t(s) = \alpha(1 - (1 - s)t)$. Note that $\alpha_1 = \alpha$ and α_0 is constant at $\alpha(1) = y_0$.

Then the function

$$H_t = \overline{\alpha_t} \cdot (h_t \circ \gamma) \cdot \alpha_t$$

defines a path-homotopy $\overline{c}_{y_0} \cdot g(\gamma) \cdot c_{y_0} \simeq_p \overline{\alpha} \cdot f(\gamma) \cdot \alpha = \Phi_\alpha(f(\gamma))$. ■

Proof of Theorem 1.11. Let $g : Y \longrightarrow X$ be a quasi-inverse to f. Then $g \circ f \simeq \mathrm{id}_X$, so Prop 1.13 gives us a diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow[\cong]{\mathrm{id}_*} & \pi_1(X, x_0) \\ & \searrow^{(gf)_*} & \cong \downarrow \Phi_\alpha \\ & & \pi_1(X, gf(x_0)) \end{array}$$

Now $(gf)_*$ must be an isomorphism since the other two maps in the diagram are isomorphisms. Since $(gf)_* = g_* \circ f_*$, the map f_* must be injective and similarly g_* must be surjective.

But now we can swap the roles of f and g, getting a diagram

$$\begin{array}{ccc} \pi_1(Y, f(x_0)) & \xrightarrow[\cong]{\mathrm{id}_*} & \pi_1(Y, f(x_0)) \\ & \searrow^{(fg)_*} & \cong \downarrow \Psi_\alpha \\ & & \pi_1(Y, fgf(x_0)) \end{array}$$

It then follows that $g_* : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, gf(x_0))$ is injective. Since we already showed it is surjective, we deduce that it is an isomorphism. Now going back to our first diagram, we get

$$g_* \circ f_* = \Phi_\alpha, \qquad \text{or} \qquad f_* = g_*^{-1} \circ \Phi_\alpha,$$

so that $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism. ■

So far, we know a number of simply connected spaces ($\mathbb{R}^n$, S^n for $n \geq 2$), and we know that $\pi_1(T^n) \cong \mathbb{Z}^n$ for any $n \geq 1$. Can there be torsion in the fundamental group? For example, is it possible that for some nontrivial loop γ in X, winding around the loop twice gives a trivial loop? The next example will have this property.

Mon, Jan. 22

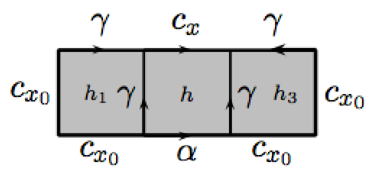
1.4. **Fundamental group of $\mathbb{RP}^2$.** Recall that the real projective plane $\mathbb{RP}^2$ is defined as the quotient of S^2 by the equivalence relation $x \sim -x$. The equivalence classes are precisely the sets of pairs of antipodal points. Another way to think about this is that each pair of antipodal points corresponds to a straight line through the origin. We will determine $\pi_1(\mathbb{RP}^2)$. Today, we're going to calculate $\pi_1(\mathbb{RP}^2)$, but first I want to discuss a result about contractibility of paths.

Proposition 1.14.

1. *Let $\alpha \in \pi_1(X, x_0)$. Then $\alpha \simeq_p c_{x_0}$ if and only if $\alpha : S^1 \longrightarrow X$ extends to a map $D^2 \longrightarrow X$.*
2. *Let α and β be paths in X from x to y. Then $\alpha \simeq_p \beta$ if and only if the loop $\alpha * \overline{\beta}$ is null.*

Proof.

1. ($\Rightarrow$) This follows from Homework II.1.
   ($\Leftarrow$) Again using Homework II.1, we may assume given a homotopy $h : \alpha \simeq c_x$. Since h is not assumed to be a path-homotopy, the formula $\gamma(s) = h(0, s)$ defines a possible nontrivai path. The picture



   where $h_1(s,t) = \gamma(st)$ and $h_3(s,t) = \overline{\gamma}(st)$, defines a path-homotopy $H : \alpha \simeq_p \gamma \cdot c_x \cdot \overline{\gamma}$.
2. The point is that
$$\alpha \simeq_p \beta \qquad \Rightarrow \qquad \alpha\overline{\beta} \simeq_p \beta\overline{\beta} \simeq_p c_x$$
   and similarly
$$\alpha\overline{\beta} \simeq_p c_x \qquad \Rightarrow \qquad \alpha \simeq_p \alpha\overline{\beta}\beta \simeq_p c_x\beta \simeq_p \beta$$

■

Recall that for S^1, the exponential map $p : \mathbb{R} \longrightarrow S^1$ was key. The analogue of that map for $\mathbb{RP}^2$ will be the quotient map
$$q : S^2 \longrightarrow \mathbb{RP}^2.$$
Note that in this case, the "fiber" (the preimage of the basepoint) consists of two points. Another ingredient that was used for S^1 was that it has a nice cover. The same is true for $\mathbb{RP}^2$: there is a cover of $\mathbb{RP}^2$ by open sets U_1, U_2, U_3 such that each preimage $q^{-1}(U_i)$ is a disjoint union $V_{i,1} \amalg V_{i,2}$ such that on each component $V_{i,j}$, the map q gives a homeomorphism $q : V_{i,j} \cong U_i$. For instance, U_1 consists of points $q(x, y, z)$ with $x \neq 0$. Then $q^{-1}(U_1)$ is the disjoint union of the left and right open hemispheres in S^2. On each hemisphere H, q restricts to a homeomorphism $q : H \cong U_1$.

For any point $x \in q^{-1}(\overline{1}) = \{-1, 1\}$, we define a loop $\Gamma(x)$ at $\overline{1}$ in $\mathbb{RP}^2$ as follows: take any path α in S^2 from 1 to x. Then $\Gamma(x) = q\alpha$ is a loop in $\mathbb{RP}^2$. Note that this is well-defined **because S^2 is simply-connected**, so that any two paths between 1 and x are homotopic. When $x = 1$, this of course gives the class of the constant loop, but when $x = -1$, this gives a nontrivial loop in $\mathbb{RP}^2$. We claim that this is a bijection. So there is only one nontrivial loop!

To see this, we construct an inverse $w : \pi_1(\mathbb{RP}^2) \longrightarrow \{-1, 1\}$. We need some lemmas:

Lemma 1.15. *Given any loop in $\mathbb{RP}^2$, there is a unique lift to a path in S^2 starting at 1.*

The proof of this lemma is **exactly the same** as that of the first lemma in the proof for the circle.

Lemma 1.16. *Let $h : \gamma \simeq_p \delta$ be a path-homotopy between loops at $\overline{1}$ in $\mathbb{RP}^2$. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow S^2$ such that $\tilde{h}(0,0) = 1$.*

Again, the proof here is identical to that for the sphere. Let's see how we can use the lemmas to define w. Given any loop γ in $\mathbb{RP}^2$, there is a unique lift $\tilde{\gamma}$ in S^2 starting at 1. Since it is a lift of a loop, we must have $\tilde{\gamma}(1) \in \{-1, 1\}$. So we define $w(\gamma) = \tilde{\gamma}(1)$. That this is well-defined follows from the second lemma.

It remains to show that w really is the inverse. Let $x \in \{-1, 1\}$. Then $\Gamma(x) = q \circ \alpha$ for some path α in S^2 from 1 to x. To compute $w(\Gamma(x))$, we must find a lift of $\Gamma(x)$, but we already know that α is the lift. Thus $w(\Gamma(x)) = \alpha(1) = x$.

Similarly, suppose γ is any loop in $\mathbb{RP}^2$. Let $\tilde{\gamma}$ be a lift. Then $\Gamma(w(\gamma)) = \Gamma(\tilde{\gamma}(1)) = q\alpha$, where α is any path from 1 to $\tilde{\gamma}(1)$. But of course $\tilde{\gamma}$ is such a path and $\gamma = q\tilde{\gamma}$.

Note that we have given a bijection between $\pi_1(\mathbb{RP}^2)$ and $\{-1, 1\}$, but we have not talked about a group structure. That's because we don't need to: there is only one group of order two! We have shown that

$$\pi_1(\mathbb{RP}^2) \cong C_2.$$

In fact, the same proof (replacing S^2 by S^n) shows that, for $n \geq 2$, we have $\pi_1(\mathbb{RP}^n) \cong C_2$.

Wed, Jan. 24

1.5. **Fundamental group of $S^1 \vee S^1$.** We will do one more example before describing the repeated phenomena we have seen in these examples. First, recall from last semester that given based spaces (X, x_0) and (Y, y_0), their **wedge sum**, or one-point union, is $X \vee Y = X \amalg Y/\sim$, where $x_0 \sim y_0$. Today, we want to study the fundamental group of $S^1 \vee S^1$ following the same approach as in the previous examples. We want to once again find a nice map $p : X \longrightarrow S^1 \vee S^1$ for some X. What we really want is an example of the following:

Definition 1.17. A surjective map $p : E \longrightarrow B$ is called a **covering map** if every $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union $p^{-1}(U) = \amalg_i V_i$ and such that, for each i, the map p restricts to a homeomorphism $p : V_i \xrightarrow{\cong} U$. We say that the neighborhood U is **evenly covered** by p.

Remark 1.18. It is common to assume that E is connected and locally path-connected. We will assume this from now on, as it simplifies the theory. So as to avoid repeatedly saying (or writing) "connected and locally path-connected", I will simply call these spaces **very connected**.

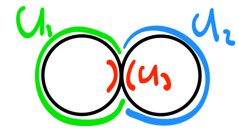
It is important to note that the neighborhood condition is local in B, not E. This contrasts with the following definition.

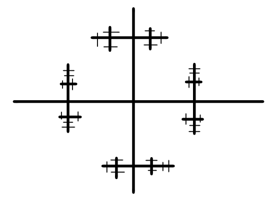
Definition 1.19. A map $f : X \longrightarrow Y$ is said to be a **local homeomorphism** if every $x \in X$ has a neighborhood U such that $f(U) \subseteq Y$ is open and $f_{|_U} : U \xrightarrow{\cong} f(U)$ is a homeomorphism.

Every covering map is a local homeomorphism: given $e \in E$, take an evenly covered neighborhood U of $p(e)$. Then e is contained in one of the V_j's, which is the desired neighborhood . The converse is not true, as the next example shows.

Example 1.20. Consider the usual exponential map $p : \mathbb{R} \longrightarrow S^1$, but now restrict it to $(0, 8.123876)$. This is a local homeomorphism but not a covering map. For instance, the standard basepoint of S^1 has no evenly covered neighborhood under this map.

Ok, now back to $S^1 \vee S^1$. It is tempting to take $X = \mathbb{R}$ since $S^1 \vee S^1$ looks locally like a line, but there is a problem spot at the crossing of the figure eight. To fix this, we might try to take X to be the union of the coordinate axes inside of $\mathbb{R}^2$. This space is really just $\mathbb{R} \vee \mathbb{R}$, and so we have the map $p \vee p : \mathbb{R} \vee \mathbb{R} \longrightarrow S^1 \vee S^1$. We want a cover of $S^1 \vee S^1$ which is nicely compatible with our map from X. Suppose we consider the cover U_1, U_2, and U_3, where U_1 is the complement of the basepoint in one circle, U_2 is the complement of the basepoint in the other, and finally U_3 is some small neighborhood of the basepoint. Well, U_1 and U_2 are good neighborhoods for $p \vee p$, but U_3 is not. The map $p \vee p$ does not give a homeomorphism from each component of the preimage of U_3 to U_3. To fix this, we would want to add infinitely many cross-sections to each of the axes.





Instead, we take X to be the fractal space given in the picture (see also page 59 of Hatcher). We define $p : X \longrightarrow S^1 \vee S^1$ as follows. On *horizontal* segments, use the exponential map to the *right* branch of $S^1 \vee S^1$. On *vertical* segments, use the left branch. Then the cover U_1, U_2, and U_3 from above is compatible with this new map p, and we see that p is a covering map.

Lemma 1.21. *The space X is simply-connected.*

Proof. The main point is that any loop in X is compact and therefore contained in a *finite* union of edges. Consider the edge furthest from the basepoint that contains part of the loop. The loop is homotopic to one constant on this furthest edge. This furthest edge is now no longer needed, and we have a new furthest edge. We can repeat until the loop is completely contracted. ∎

Let $F = p^{-1}(*)$ be the fiber. Any point in this fiber may be uniquely described as a "word" in the letters u, r, d, and l. Define

$$\Gamma : F \longrightarrow \pi_1(S^1 \vee S^1)$$

as follows: given $y \in F$, let α_y be any path in X from the basepoint to y. Then $\Gamma(y) = p \circ \alpha$. It does not matter which α_y we choose since X is simply-connected. We will define an inverse to Γ, but we now state the needed lemmas in the generality of coverings.

Lemma 1.22. *Let $p : E \longrightarrow B$ be a covering and suppose $p(e) = b$. Given any path starting at b in B, there is a unique lift to a path in E starting at e.*

The proof of this lemma is **exactly the same** as that of Lemma 1.3, for the circle.

Lemma 1.23. *Let $p : E \longrightarrow B$ be a covering and suppose $p(e) = b$. Let $h : \gamma \simeq_p \delta$ be a path-homotopy between paths starting at b in B. Then there is a unique lift $\tilde{h} : I \times I \longrightarrow E$ such that $\tilde{h}(0,0) = e$.*

Just as in the previous examples, the above lemmas allow us to define $w : \pi_1(S^1 \vee S^1) \longrightarrow F$ by the formula $w(\gamma) = \tilde{\gamma}(1)$. We will skip the verification that Γ and w are inverse, as this really follows the same script.

Fri, Jan. 26

We have established a bijection between $\pi_1(S^1 \vee S^1)$ and the set of "words" in the letters u, r, d, and l. It remains to describe the group structure. For this, we will back up a little.

Definition 1.24. Let $p : E \longrightarrow B$ and $q : E' \longrightarrow B$ be covers of a space B. A **map of covers** from E to E' is simply a map of spaces $\varphi : E \longrightarrow E'$ such that $q \circ f = p$. These are also sometimes called **covering homomorphisms**.

The special case in which the two covers are the *same* cover and f is a homeomorphism is referred to as a **deck transformation**. We write $\mathrm{Aut}(E)$ for the set of all deck transformations of E. This is a group under composition.

Keeping our notation from earlier, let $b \in B$ be a basepoint and write $F = p^{-1}(b)$ for the fiber. Note that any deck transformation $\varphi : E \longrightarrow E$ must take F to F. Let us pick a basepoint e for E. Since we want the covering map q to be based, this means that e lies in the fiber F. We may now define a map $A : \mathrm{Aut}(E) \longrightarrow F$ by $A(\varphi) = \varphi(e)$.

Theorem 1.25. *Let $p : X \longrightarrow B$ be a covering such that X is simply connected. Then the map $A : \mathrm{Aut}(X) \longrightarrow F$ is a bijection and the composition $\Gamma \circ A$ is an isomorphism of groups $\mathrm{Aut}(X) \cong \pi_1(B)$.*

Proof. Let us first show that A is injective. Thus let φ_1 and φ_2 be deck transformations which agree at e. Let $x \in X$ be any point and let α be any path in X from e to x. Then the paths $\varphi_1 \circ \alpha$ and $\varphi_2 \circ \alpha$ are both lifts of $p \circ \alpha$ starting at the common point $\varphi_1(e) = \varphi_2(e)$. By the uniqueness of lifts, these must be the same path. It follows that their endpoints, $\varphi_1(x)$ and $\varphi_2(x)$ agree.

It remains to show that A is surjective. Let $f \in F$ be any point in the fiber. We wish to produce a deck transformation $\varphi : X \longrightarrow X$ such that $\varphi(e) = f$. We build the map φ locally and patch together. Let $x \in X$ and pick any path $\alpha : e \rightsquigarrow x$. Then $p\alpha$ is a path in B starting at b and ending at px. By the path-lifting lemma, there is a unique lift $\widetilde{p\alpha}$ in X starting at f. We define $\varphi(x) = \widetilde{p\alpha}(1)$. From this definition, continuity is not at all clear. But the point is that since p is a covering, we can choose an evenly-covered neighborhood U of $p(x)$. Let V be the slice of $p^{-1}(U)$ containing x and V' the slice containing $\varphi(x) = \widetilde{p\alpha}(1)$. Then the restriction of φ to V is the composition of homeomorphisms

$$V \xrightarrow{p} U \xleftarrow{p} V'.$$

By the local criterion for continuity (Prop 2.19 in Lee), it follows that φ is continuous.

By construction, φ will be a map of covers, as long as we can verify that it is well-defined. But if $\delta : e \rightsquigarrow x$ is another choice of path, we know that $\alpha \simeq_p \delta$ because X is simply-connected. It follows that $p\alpha \simeq_p p\delta$, and by lifting the path-homotopy, it follows that $\widetilde{p\alpha} \simeq_p \widetilde{p\delta}$, so that their right endpoints agree.

We will finish the proof next time.

Mon, Jan. 29

So given $f \in F$, we have built a map of covers $\varphi : X \longrightarrow X$, but we wanted this to be an isomorphism. From the construction of φ, we see that it is a local homeomorphism, which implies that it is open. Suppose $\varphi(x_1) = \varphi(x_2)$. Note that since φ is a map of covers, this implies that x_1 and x_2 are in the same fiber. Let $\alpha_1 : e \rightsquigarrow x_1$ and $\alpha_2 : e \rightsquigarrow x_2$ be paths. By hypothesis, $\widetilde{p\alpha_1}$ and $\widetilde{p\alpha_2}$ have the same endpoints. Since X is simply-connected, we know that $\widetilde{p\alpha_1} \simeq_p \widetilde{p\alpha_2}$. It follows that $p\alpha_1 \simeq_p p\alpha_2$, and it then follows, by lifting the homotopy, that $\alpha_1 \simeq_p \alpha_2$. In particular, $\alpha_1(1) = \alpha_2(1)$, so $x_1 = x_2$. This shows that φ is injective.

To see that φ is surjective, let $x \in X$. We can then pick a path $\gamma : f \rightsquigarrow x$. Then $p\gamma$ is a path in B from b to $p(x)$, which lifts uniquely to a path $\tilde{\gamma}$ from e to some point x'. But then $\varphi(x') = x$ by the definition of φ.

We have now established that

$$A : \mathrm{Aut}(X) \longrightarrow F$$

is a bijection. We also wanted to show that the resulting bijection $\Gamma \circ A : \mathrm{Aut}(X) \longrightarrow \pi_1(B)$ is a group isomorphism. It remains only to show that this is a group homomorphism.

Let $\varphi_1, \varphi_2 \in \mathrm{Aut}(X)$. Recall that $\Gamma(A(\varphi_1))$ is defined as follows: pick any path α_1 in X from e to $f_1 = \varphi_1(e)$. Then $\Gamma(A(\varphi_1)) = p \circ \alpha_1$. Similarly $\Gamma(A(\varphi_2)) = p \circ \alpha_2$. Now $A(\varphi_2 \circ \varphi_1) = \varphi_2 \circ \varphi_1(e) = \varphi_2(f_1)$. To compute Γ of this point, we need a path in X from e to $\varphi_2(f_1)$. But $\alpha_2 * \varphi_2(\alpha_1)$ is such a path. Then

$$\begin{aligned}\Gamma(A(\varphi_2 \circ \varphi_1)) = \Gamma(\varphi_2(f_1)) = p \circ (\alpha_2 * \varphi_2(\alpha_1)) &= (p \circ \alpha_2) * (p \circ \varphi_2 \circ \alpha_1) \\ &= (p \circ \alpha_2) * (p \circ \alpha_1) = \Gamma(A(\varphi_2)) * \Gamma(A(\varphi_1)).\end{aligned}$$

■

Returning now to our example $X \longrightarrow S^1 \vee S^1$, we have identified $\pi_1(S^1 \vee S^1)$ with the group of deck transformations $X \cong X$, and we know we have one such deck transformation for each point in the fiber F. Any transformation can be thought of as a sequence of horizontal and vertical "moves". Writing u for an upwards shift and r for a shift to the right, any element of the group can be described by a seqence of u's, r's, and their inverses.

Definition 1.26. A **word** in letters u, r, and their inverses is simply a sequence of these letters. We say the word is **reduced** if no u^{-1} is adjacent to a u, and similarly for the r's. The **free group** F_2 or $F(u, r)$ on the letters u and r is the set of reduced (including empty) words, where the group operation is concatenation. The inverse of any word is the same word in reversed order and with the sign of each letter reversed.

We have shown that $\pi_1(S^1 \vee S^1)$ is the free group on two letters. In particular, this is our first example of a nonabelian fundamental group.

Wed, Jan. 31

2. The theory of covering spaces

2.1. **Lifting Lemmas.** So far, the only kind of coverings we have studied have been those in which the covering space is simply connected. Now we will relax this condition and discuss the more general theory.

Proposition 2.1. *Let $p : E \longrightarrow B$ be a covering. Then the induced map $p_* : \pi_1(E) \longrightarrow \pi_1(B)$ is injective.*

Proof. Let $\gamma \in \pi_1(E)$ and suppose $p_*(\gamma) = 0$. In other words, the loop $p \circ \gamma$ in B is null. Let $h : I \times I \longrightarrow B$ be a null-homotopy. Then this lifts to a homotopy $\tilde{h} : I \times I \longrightarrow E$ from γ (the unique lift of $p \circ \gamma$) to a lift $\tilde{c}$ of the constant loop. Since the constant loop at e is a lift of the constant loop at b, uniqueness of lifts implies that $\tilde{c}$ is the constant loop. So $\tilde{h}$ is a null-homotopy for γ. ■

Example 2.2. The only example of a covering we have discussed thus far in which the covering space is not simply connected is the n-fold cover $S^1 \longrightarrow S^1$. In this case, the cover sends the generator of $\pi_1(S^1) \cong \mathbb{Z}$ to n times the generator, and the image of p_* is the subgroup $n\mathbb{Z} < \mathbb{Z}$.

Given the above result, any covering of B gives rise to a subgroup of $\pi_1(B)$. One might wonder what subgroups can arise in this way. We will see that, under mild hypotheses on B, every subgroup arises in this way.

Previously, we have studied lifting paths and path-homotopies against a covering. We can also generalize this to consider lifting arbitrary maps $f : Z \longrightarrow B$. As in Remark 1.18, whenever we discuss a covering map $E \longrightarrow B$, we assume that E is "very connected", which implies the same for B. In particular, this is assumed for the following results.

Proposition 2.3. *(Homotopy lifting) Let Z be a locally connected space. Let $p : E \longrightarrow B$ be a covering and $h : Z \times I \longrightarrow B$ be a homotopy between maps $f, g : Z \rightrightarrows B$. Let $\tilde{f}$ be a lift of f. Then there is a unique lift of h to $\tilde{h}$ with $\tilde{h}_0 = \tilde{f}$.*

Proposition 2.4. *(Unique lifting) Let $p : E \longrightarrow B$ be a covering and $f : Z \longrightarrow B$ a map, with Z connected. If $\tilde{f}$ and $\hat{f}$ are both lifts of f that agree at some point of Z, then they are the same lift.*

Note that in the second result, we are not asserting that a lift exists! See Theorems 8.3 and 8.4 of [Lee] for complete proofs.

Here is a sketch of Proposition 2.4.

Sketch. The idea is to show that the subset of Z on which the lifts agree is both open and closed; it is already given to be nonempty. For any $z \in Z$, pick an evenly-covered neighborhood U of $f(z)$. On the one hand, suppose $\tilde{f}(z) = \hat{f}(z)$. Then let V be the component of $p^{-1}(U)$ containing this point. Then $\tilde{f}^{-1}(V) \cap \hat{f}^{-1}(V)$ is a neighborhood of z on which the lifts agree (since $q : V \longrightarrow U$ is a homeomorphism).

On the other hand, if $\tilde{f}(z) \neq \hat{f}(z)$, then let $\tilde{V}$ and $\hat{V}$ be the components of $\tilde{f}(z)$ and $\hat{f}(z)$ in $p^{-1}(U)$. It follows that $\tilde{f}^{-1}(\tilde{V}) \cap \hat{f}^{-1}(\hat{V})$ is a neighborhood of z on which $\tilde{f}$ and $\hat{f}$ disagree (they land in different components of $p^{-1}(U)$). ■

Fri, Feb. 2

The interesting, new result here concerns the existence of lifts.

Proposition 2.5. *(Lifting Criterion) Let $p : E \longrightarrow B$ be a covering and let $f : Z \longrightarrow B$, with Z very connected. Given points $z_0 \in Z$ and $e_0 \in E$ with $f(z_0) = p(e_0)$, there is a lift $\tilde{f}$ with $\tilde{f}(z_0) = e_0$ if and only if $f_*(\pi_1(Z, z_0)) \subseteq p_*(\pi_1(E, e_0))$.*

Proof. ($\Rightarrow$) Since $f = p \circ \tilde{f}$, we have $f_* = p_* \circ \tilde{f}_*$.

($\Leftarrow$) Here is the more interesting direction. Suppose that $f_*(\pi_1(Z, z_0)) \subseteq p_*(\pi_1(E, e_0))$. Let $z \in Z$. We wish to define $\tilde{f}(z)$. Pick any path α in Z from z_0 to z. Then $f \circ \alpha$ is a path in B, which therefore lifts uniquely to a path $\tilde{\alpha}$ in E starting at, say e_0. We define $\tilde{f}(z) = \tilde{\alpha}(1)$. Then $\tilde{f}$ is a lift of f.

Why is the lift $\tilde{f}$ well-defined? Suppose β is another path in Z from z_0 to z. Then $f \circ (\alpha \cdot \overline{\beta})$ is a loop in B at $b_0 = f(z_0)$. By assumption, this means that for some loop δ in E, we have

$$p \circ \delta \simeq_p f \circ (\alpha \cdot \overline{\beta}) = f(\alpha) \cdot \overline{f(\beta)}$$

in B. Since path-composition behaves well with respect to path-homotopy, we have a path-homotopy

$$h : (p \circ \delta) \cdot f(\beta) \simeq_p f(\alpha)$$

of paths in B. Note that the path $(p \circ \delta) \cdot f(\beta)$ lifts to the path $\delta \cdot \tilde{\beta}$. The homotopy h then lifts (uniquely) to a path-homotopy in E

$$\tilde{h} : \delta \cdot \tilde{\beta} \simeq_p \tilde{\alpha}.$$

In particular, these have the same endpoints. Of course, the endpoint of $\delta \cdot \tilde{\beta}$ is simply the endpoint of $\tilde{\beta}$. It follows that $\tilde{f}$ is well-defined at z.

Just for emphasis, let's go through the proof that $\tilde{f}$ is continuous. Let $z \in Z$ and let U be an evenly covered neighborhood U of $f(z)$, and let V be the component of $p^{-1}(U)$ containing the lift $\tilde{f}(z)$. Let $W \subseteq Z$ be the path-component of $f^{-1}(U)$ containing z. Since Z is locally path-connected, W is open. Moreover, since W is path-connected and $\tilde{f}(W) \cap V \neq \emptyset$, we must have $\tilde{f}(W) \subseteq V$. Then on the neighborhood W of z, the lift $\tilde{f}$ may be described as the composition $p|_V^{-1} \circ f$. It follows that $\tilde{f}$ is continuous on the neighborhood W of z. Since z was arbitrary, $\tilde{f}$ is continuous. ■

This implies what we already know: S^1 is not a retract of $\mathbb{R}$. More generally, and less trivially, we have that the identity map $S^1 \longrightarrow S^1$ does not lift against the n-fold cover $p_n : S^1 \longrightarrow S^1$. Even more generally, we might ask about lifting some $p_k : S^1 \longrightarrow S^1$ against the cover $p_n : S^1 \longrightarrow S^1$. By the result above, this happens if and only if $k\mathbb{Z} \subseteq n\mathbb{Z}$. In other words, this happens if and only if $n \mid k$.

More interestingly, we have

Corollary 2.6. *Suppose that the covering space E is simply-connected. Then a map $f : Z \longrightarrow B$ lifts to some $\tilde{f} : Z \longrightarrow E$ if and only if f induces the trivial map on fundamental groups.*

Corollary 2.7. *Suppose that Z is simply-connected and $p : E \longrightarrow B$ is a covering map. Then any map $f : Z \longrightarrow B$ lifts against p.*

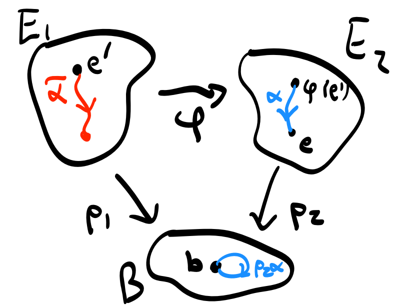
Thus if $X \longrightarrow B$ is a simply connected covering and $E \longrightarrow B$ is any covering, we automatically get a map of covers $X \longrightarrow E$. For this reason, simply connected covers are referred to as **universal covers**.

Mon, Feb. 5

Proposition 2.8. *Suppose that $\varphi : E_1 \longrightarrow E_2$ is a map of covers. Then φ is a covering map.*

Proof. We start by showing that φ is surjective. Let $e \in E_2$. Let $b = p_2(e)$, and pick any $e' \in p_1^{-1}(b)$. Since E_2 is very connected, we can find a path $\alpha : \varphi(e') \rightsquigarrow e$ in E_2. We can push this path α down to a loop $p_2\alpha$ in B and then lift this uniquely to a path $\tilde{\alpha}$ in E_1 starting at e'. Now $\varphi(\tilde{\alpha})$ is a lift of $p_2\alpha$ in E_2 starting at $\varphi(e')$, so by uniqueness of lifts, we must have $\varphi(\tilde{\alpha}) = \alpha$. In particular, $\varphi(\tilde{\alpha}(1)) = e$.

Now we show that e has an evenly-covered neighborhood of e. We know that the point $p_2(e) \in B$ has an evenly covered neighborhood U_2 (with respect to p_2). Let U_1 be an evenly covered neighborhood, with respect to p_1, of $p_2(e)$. Write U for the component of $U_1 \cap U_2$ containing $p_2(e)$. Then $p_2^{-1}(U) \cong \amalg V_i$. Let V_0 be the component containing e. Write $p_1^{-1}(U) \cong \amalg W_j$. Then, since U is connected, each V_i and W_j must be connected. It follows that φ takes each W_j into a single V_i, so that $\varphi^{-1}(V_0) \subseteq p_1^{-1}(U)$ is a disjoint union of some of the W_j's, and it follows that φ restricts to a homeomorphism on each component because both p_1 and p_2 do so.



■

It follows that any universal cover $X \longrightarrow B$ *covers every other covering* $E \longrightarrow B$.

Remark 2.9. Recall that in the proof of Theorem 1.25, we ended up building a map of covers $\varphi : X \longrightarrow X$ corresponding to any point in the fiber F, but we wanted to know it was in fact a homeomorphism. Prop 2.8 now gives us that it is a covering map, so that according to the homework, it suffices to show that the φ we constructed was injective. This can be seen by verifying that it is injective on each fiber.

2.2. **The monodromy action.**

Our next goal is to completely understand the possible covers of a given space B. There are two avenues of approach. On the one hand, Prop. 2.1 tells us that covering spaces give rise to subgroups of $\pi_1(B)$, so we can try to understand the collection of subgroups. Another approach, which we will look at next, focuses on the fiber $F = p^{-1}(b_0)$.

It will be convenient in what follows to write $G = \pi_1(B, b_0)$ and $F = p^{-1}(b_0) \subset E$. Given a loop γ based at b_0 and a point $f \in F$, we will write $\tilde{\gamma}_f$ for the lift of γ which starts at f.

Theorem 2.10. *Let $p : E \longrightarrow B$ be a covering and let $F = p^{-1}(b)$ be the fiber over the basepoint. Then the function*

$$a : F \times \pi_1(B) \longrightarrow F, \qquad (f, [\gamma]) \mapsto \tilde{\gamma}_f(1)$$

specifies a transitive right action of $\pi_1(B)$ on the fiber F. This is called the **monodromy action**.

Proof. Recall that we have already showed this to be well-defined.

Let c_{b_0} be the constant loop at b_0. Then the constant loop c_f at f in E is a lift of c_{b_0} starting at f, so by uniqueness it must be the only lift. Thus $f \cdot [c_{b_0}] = f$.

Now let α and β be loops at b. We wish to show that $(f \cdot \alpha) \cdot \beta = f \cdot (\alpha \cdot \beta)$. Let $f_2 = \tilde{\alpha}_f(1)$. Then $\tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}$ is a (= the) lift of $\alpha \cdot \beta$ starting at f, so

$$f \cdot (\alpha \cdot \beta) = \tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}(1).$$

On the other hand, $f \cdot \alpha = \tilde{\alpha}_f(1) = f_2$, so

$$(f \cdot \alpha) \cdot \beta = f_2 \cdot \beta = \tilde{\beta}_{f_2}(1)$$

Finally, to see that this action is transitive, let f_1 and f_2 be points in the fiber F. Let γ be a path in E from f_1 to f_2. Then $\alpha = p \circ \gamma$ is a loop at b_0. Furthermore $\tilde{\alpha}_{f_1} = \gamma$, so $f_1 \cdot \alpha = \gamma(1) = f_2$. ■

Note that if we instead wrote path-composition in the "correct" order (i.e. in the same order as function composition), this would give a left action of $\pi_1(B)$ on F.

By the Orbit-Stabilizer theorem, since G acts transitively on F, there is an isomorphism of right G-sets $F \cong G_{e_0} \backslash G$, where $G_{e_0} \leq G$ is the stabilizer of e_0.

Wed, Feb. 7

Proposition 2.11. *The stabilizer of $e \in F$ under the monodromy action is the subgroup $p_*(\pi_1(E, e)) \leq \pi_1(B, b_0)$.*

Proof. Let $[\gamma] \in \pi_1(E, e)$. Then γ is a lift of $p \circ \gamma$ starting at e, so $e \cdot p_*(\gamma) = \gamma(1) = e$. Thus $p_*(\gamma)$ stabilizes e.

On the other hand, let $[\alpha] \in \pi_1(B, b_0)$ and suppose that $e \cdot [\alpha] = e$. This means that α lifts to a loop $\tilde{\alpha}$ in E. Thus $\alpha = p \circ \tilde{\alpha}$ and $[\alpha] \in p_*(\pi_1(E, e))$. ■

Corollary 2.12. *Let $p : E \longrightarrow B$ be a covering. Then, writing $H = p_*(\pi_1(E, e))$ the map*

$$H\backslash\pi_1(B, b) \xrightarrow{\cong} F.$$
$$H\,\gamma \mapsto f \cdot \gamma$$

is an identification of right $\pi_1(B)$-sets

We have seen that any covering gives rise to a transitive G-set. We would also like to understand maps of coverings.

Definition 2.13. Let X and Y be (right) G-sets. A function $f : X \longrightarrow Y$ is said to be **G-equivariant** (or a map of G-sets) if $f(xg) = f(x) \cdot g$ for all x.

Proposition 2.14. *Let $\varphi : E_1 \longrightarrow E_2$ be a map of covers of B. The induced map on fibers $F_1 \longrightarrow F_2$ is $\pi_1(B)$-equivariant.*

Proof. Let $[\gamma] \in \pi_1(B)$ and $f \in F_1$. We have $f \cdot [\gamma] = \tilde{\gamma}_f(1)$, where $\tilde{\gamma}_f$ is the lift of γ starting at f. Similarly, we have $\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1)$. But $\varphi(\tilde{\gamma})$ is a lift of γ starting at $\varphi(\gamma(0)) = \varphi(f)$, so $\tilde{\gamma}_{\varphi(f)} = \varphi(\tilde{\gamma}_f)$. Thus

$$\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1) = \varphi(\tilde{\gamma}_f)(1) = \varphi(\tilde{\gamma}_f(1)) = \varphi(f \cdot [\gamma]).$$

■

Proposition 2.15. *Let $H, K \leq G$. Then every G-equivariant map $\varphi : H\backslash G \longrightarrow K\backslash G$ is of the form $Hg \mapsto K\gamma g$ for some $\gamma \in G$ satisfying $\gamma H \gamma^{-1} \leq K$.*

Proof. Since $H\backslash G$ is a transitive G-set, an equivariant map out of it is determined by the value at any point. Suppose we stipulate

$$He \mapsto K\gamma.$$

Then equivariance would force

$$Hg \mapsto K\gamma g.$$

Is this well-defined? Since $Hg = Hhg$ for any $h \in H$, we would need $K\gamma g = K\gamma hg$. Multiplying by $g^{-1}\gamma^{-1}$ gives $K = K\gamma h \gamma^{-1}$. Since $h \in H$ is arbitrary, this says that $\gamma H \gamma^{-1} \leq K$. ■

Corollary 2.16. *A G-equivariant map $\varphi : H\backslash G \longrightarrow K\backslash G$ exists if and only if H is conjugate in G to a subgroup of K. The two orbits are isomorphic (as right G-sets) if and only if H is conjugate to K.*

Notation. Given covers (E_1, p_1) and (E_2, p_2) of B, we denote by $\mathrm{Map}_B(E_1, E_2)$ the set of covering homomorphisms $\varphi : E_1 \longrightarrow E_2$. Given two right G-sets X and Y, we denote by $\mathrm{Hom}_G(X, Y)$ the set of G-equivariant maps $X \longrightarrow Y$.

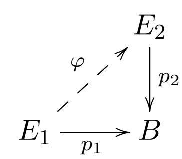
The following theorem classifies covering homomorphisms.

Theorem 2.17. *Let E_1 and E_2 be coverings of B. Then Proposition 2.14 induces a bijection*

$$\mathrm{Map}_B(E_1, E_2) \xrightarrow{\cong} \mathrm{Hom}_G(F_1, F_2).$$

Proof. The key is that a covering homomorphism is a lift in the diagram to the right. Uniqueness of lifts gives injectivity in the theorem. For surjectivity, we use the lifting criterion Prop 2.5. Thus suppose given a G-equivariant map $\lambda : F_1 \longrightarrow F_2$ and fix a point $e_1 \in F_1$. Let $e_2 = \lambda(e_1) \in F_2$. The lifting criterion will provide a lift if we can verify that

$$(p_1)_*(\pi_1(E_1, e_1)) \leq (p_2)_*(\pi_1(E_2, e_2)).$$



But remember that according to Prop 2.11, these are precisely the stabilizers of e_1 and e_2, respectively. Writing H_1 and H_2 for these groups, the map $\lambda : F_1 \longrightarrow F_2$ corresponds to a map

$$\widehat{\lambda} : H_1\backslash G \longrightarrow H_2\backslash G.$$

According to Prop 2.15, this means that $\gamma H_1 \gamma^{-1} \leq H_2$, where $\widehat{\lambda}(H_1 e) = H_2\gamma$. The fact that $\lambda(e_1) = e_2$ means that $\gamma = e$. So $H_1 \leq H_2$ as desired. ∎

Corollary 2.18. *If E is a cover of B, then we have group isomorphisms*

$$\mathrm{Aut}_B(E) \cong \mathrm{Aut}_G(H\backslash G, H\backslash G) \cong N_G(H)/H,$$

where $N_G(H)$ is the **normalizer** *of H in G, consisting of those elements of G which conjugate H to itself.*

Proof. Theorem 2.17 gives the first bijection. By Corollary 2.15, we have a surjective group homomorphism $N_G(H) \longrightarrow \mathrm{Aut}_G(H\backslash G, H\backslash G)$, and it remains only to identify the kernel. But $\gamma \in N_G(H)$ lies in the kernel if $Hg \mapsto H\gamma g$ is the identity map of $H\backslash G$, which happens just if $\gamma \in H$. So we conclude that the kernel is H. ∎

The quotient group $N_G(H)/H$ is known as the **Weyl group** of H in G and is sometimes denoted $W_G(H)$.

Fri, Feb. 9

2.3. **The classification of covers.** We have almost shown that working with covers of B is the same as working with transitive right G-sets (technically, we are heading to an "equivalence of categories"). All that is left is to show that for every G-orbit F, there is a cover $p : E \longrightarrow B$ whose fiber is F as a G-set.

We assume that B has a universal cover $q : X \longrightarrow B$. Recall that we showed in Theorem 1.25 that the group of deck transformations of X is isomorphic to G.

Proposition 2.19. *The (left) action of G on X via deck transformations is free and properly discontinuous.*

Proof. Let $x \in X$ and suppose $gx = x$ for some $g \in G$. Recall that here g is a covering homomorphism $X \longrightarrow X$ and thus a lift of $q : X \longrightarrow B$. By the uniqueness of lifts, since g looks like the identity at the point x, it must be the identity. This shows the action is free.

Again, let $x \in X$. We want to find a neighborhood V of x such that only finitely many translates gV meet V. Consider $b = q(x)$. Let U be an evenly-covered neighborhood of b. Then $q^{-1}(U) \cong \coprod V_i$, and $x \in Vj$ for some j. Recall that G freely permutes the pancakes V_i. In particular, the only translate of V_j that meets V_j is the identity translate eV_j. ■

According to Homework IV.2, this means that the quotient map $X \longrightarrow G\backslash X$ is a cover. Actually, the cover $X \xrightarrow{q} B$ factors through a homeomorphism $G\backslash X \cong B$. If we consider the action of a subgroup $H \leq G$, it is still free and properly discontinuous. So we get a covering

$$q_H : X \longrightarrow H\backslash X = X_H$$

for every H. Moreover, the universal property of quotients gives an induced map

$$p_H : H\backslash X \longrightarrow B.$$

Proposition 2.20. *The map $p_H : H\backslash X \longrightarrow B$ is a covering map, and the fiber F is isomorphic to $H\backslash G$ as a G-set.*

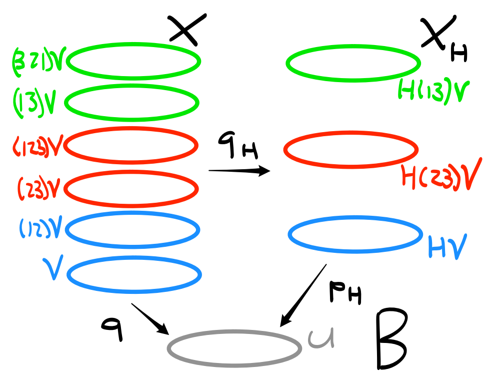
Proof. Let $b \in B$. Then we have a neighborhood U which is evenly-covered by q. Recall again that the G-action, and therefore also the H-action, simply permutes the pancakes in $p^{-1}(U)$. We thus get an action of H on the indexing set $\mathcal{I}$ for the pancakes in $p^{-1}(U)$. If we write $W_i = q_H(V_i)$, we thus have the diagram

$$\begin{array}{ccccc} q^{-1}(U) & \xrightarrow{q_H} & p_H^{-1}(U) & \xrightarrow{p_H} & U \\ \uparrow \cong & & \uparrow \cong & & \| \\ \coprod_{i \in \mathcal{I}} V_i & \longrightarrow & \coprod_{j \in H\backslash\mathcal{I}} W_j & \longrightarrow & U \end{array}$$

To see that the restriction of p_H to a single W_j gives a homeomorphism, we use the fact that $q_H : V_j \longrightarrow W_j$ is a homeomorphism, since $q_H : X \longrightarrow X_H$ is a covering, and that $q : V_j \longrightarrow U$ is a homeomorphism. It follows that $p_H = q \circ q_H^{-1}$ is a homeomorphism.

For the identification of the fiber $F \subseteq X_H$, notice that the H-action on X acts on each fiber separately, and the quotient of this action on the fiber of X gives precisely $H\backslash G$. ■

Example 2.21. Suppose that $G = \Sigma_3$, the symmetric group on 3 letters, and let $H = \{e, (12)\} \leq G$. If we take an evenly-covered neighborhood U in B, then the situation described in the proof above is given in the picture to the right.
As an aside, note that X_H here is an example of a covering in which the deck transformations do *not* act transitively on the fibers.



To sum up, we have shown that *if* B has a universal cover, then the assignment $(E, p) \mapsto F$ gives an "equivalence of categories" between coverings of B (Cov_B) and G-orbits (Orb_G).

Mon, Feb. 12

We can form a category Cov_B whose objects are the covers of B and whose morphisms are the maps of covers. We can also form a category Orb_G whose objects are the transitive (right) G-sets. Our recent discussion has shown that the assignment (technically 'functor')

$$\mathrm{Cov}_B \longrightarrow \mathrm{Orb}_G, \qquad (E,p) \mapsto F := p^{-1}(b_0)$$

is an equivalence of categories. This means that

(1) (fully faithful) We have a bijection $\mathrm{Cov}_B(E,E') \cong \mathrm{Orb}_G(F,F')$
(2) (essentially surjective) Every G-orbit arises in this way, meaning that any G-orbit is isomorphic to $p^{-1}(b_0)$ for some cover of B.

One consequence of having an equivalence of categories is that this produces a bijection between isomorphism classes of objects.

Corollary 2.22. *The fiber functor* $\mathrm{Cov}_B \longrightarrow \mathrm{Orb}_G$ *induces a bijection*

$$\{\textit{isomorphism classes of covers}\} \cong \{\textit{isomorphism classes of orbits}\}$$
$$\cong \{\textit{conjugacy classes of subgroups of } G\}$$

Note that there is no obvious choice of functor in the other direction. Given a G-orbit X, picking a point in the orbit produces an isomorphism to some $H\backslash G$, and then Proposition 2.20 produces a cover whose fiber is isomorphic to X. But this really does involve making a choice. This is a pretty typical situation: a functor that is essentially surjective and fully faithful is called an equivalence of categories, but to produce a functor that looks like an inverse, choices need to be made.

2.4. **Existence of universal covers.** The last result we need to tie this story together is the existence of universal covers.

Definition 2.23. Let B be any space. A subset $U \subseteq B$ is **relatively simply connected** (in B) if every loop in U is contractible in B. We say that B is **semilocally simply connected** if every point has a relatively simply connected neighborhood.

Remark 2.24. Note that if B is very connected and semilocally simply connected, then every point has a path-connected, relatively simply connected neighborhood. This is because if $b\, inU$ is relatively simply connected, then the path component of b in U is open (B is locally path-connected) and also relatively simply-connected (true of any subset of a relatively simply connected subset).

Theorem 2.25. *Let B be very connected. Then there exists a universal cover $X \longrightarrow B$ if and only if B is semilocally simply connected.*

♡♡♡ **Wed, Feb. 14** ♡♡♡

Proof. The forward implication is left as an exercise. For convenience, we fix a basepoint $b_0 \in B$.

We start by working backwards. That is, suppose that $q : X \longrightarrow B$ exists. Given a point $b \in B$, what can we say about the fiber $q^{-1}(b)$? Pick a basepoint $x_0 \in q^{-1}(b_0)$. Then, for each $f \in q^{-1}(b)$, we get a (unique) path-homotopy class of paths $\alpha : x_0 \rightsquigarrow f$. Composing with the covering map q gives a (unique) path-homotopy class of paths $q \circ \alpha : b_0 \rightsquigarrow b$. This now gives a description of the fiber $q^{-1}(b)$ purely in terms of B.

We now take this as a starting point. As a set, we take X to be the set of path-homotopy classes of paths starting at b_0. The map $q : X \longrightarrow B$ takes a class $[\gamma]$ to the endpoint $\gamma(1)$. It remains to (1) topologize X, (2) show that q is a covering map, and (3) show that X is simply-connected.

We specify the topology on X by giving a basis. Let γ be a path in B starting at b_0. Let U be any path-connected, relatively simply-connected neighborhood of the endpoint $\gamma(1)$. Define a subset $U[\gamma] \subseteq X$ to be the set of equivalence classes of paths of the form $[\gamma\delta]$, where $\delta : I \longrightarrow U$ is a path in U. These cover X since each $[\gamma]$ is contained in some $U[\gamma]$ by Remark 2.24. Now suppose that $\gamma \in U_1[\gamma_1] \cap U_2[\gamma_2]$. Then the path-component of $\gamma(1)$ in $U_1 \cap U_2$ is again path-connected and relatively simply connected. Thus

$$\gamma \in U[\gamma] \subseteq U_1[\gamma_1] \cap U_2[\gamma_2].$$

We have shown that the $U[\gamma]$ give a basis for a topology on X.

Next, we show that q is continous. Let $V \subseteq B$ be open and let $q([\gamma]) \in V$, so that $\gamma(1) \in V$. Then we can find a path-connected, relatively simply connected U satisfying $\gamma(1) \in U \subseteq V$. So $U[\gamma]$ is a neighborhood of $[\gamma]$ in $q^{-1}(V)$, as desired.

Since B is path-connected, it follows that q is surjective. Let $b \in B$ and let $b \in U$ be a path-connected, relatively simply-connected neighborhood. We claim that U is evenly covered by q. First, we claim that

$$q^{-1}(U) = \bigcup_{[\gamma] \in q^{-1}(b)} U[\gamma].$$

It is clear that the RHS is contained in the LHS. Suppose that $q([\alpha]) \subseteq U$. Then $\alpha(1) \in U$ and we may pick a path $\delta : \alpha(1) \rightsquigarrow b$ in U. Then $\alpha \in U[\alpha\delta]$.

(This will be continued on Monday . . .) ■

Fri, Feb. 16

Exam day!!

Mon, Feb. 19

Proof. (Continued ...) Finally, we wish to show that this is a disjoint union. By the definition of the topology on X, each $U[\gamma]$ is open. Thus suppose that $[\alpha] \in U[\gamma_1] \cap U[\gamma_2]$. This means that

$$[\alpha] = [\gamma_1 \delta_1] = [\gamma_2 \delta_2].$$

In other words,

$$[\gamma_1 \delta_1 \overline{\delta_2}] = [\gamma_2].$$

Since U is relatively simply-connected, this implies that $[\gamma_1] = [\gamma_2]$. So any two overlapping $U[\gamma]$ are in fact the same. To finish the proof that q is a covering, we need to show that q restricts to a homeomorphism $q : U[\gamma] \xrightarrow{\cong} U$. Surjectivity follows from the assumption that U is path-connected. Injectivity is the relatively simply-connected hypothesis. Finally, q takes any basis $V[\lambda]$ to the open set V (since V is path-connected), so it is open. We have shown that q is a covering map.

The final step is to show that X is very connected and simply connected. Since X is locally homeomorphic to B and B is locally path-connected, it follows that the same is true of X. Next, we show that X is path-connected (and therefore connected). Let $[\gamma] \in X$. We define a path h in X from the constant path $[c_{b_0}]$ to $[\gamma]$ by $h(s) = [\gamma|_{[0,s]}]$. In the interest of time, we skip the verification that h is continuous (but see Lee, proof of Theorem 11.43).

To see that X is simply connected, let Γ be a loop in X at the basepoint $[c_{b_0}]$. Write $\gamma = q \circ \Gamma$. Then Γ is a lift of γ, but so is the loop $s \mapsto [\gamma_{[0,s]}]$. By uniqueness of lifts, $[\Gamma(s)] = [\gamma_{[0,s]}]$. Then, since Γ is a loop, we have

$$[\gamma] = [\gamma_{[0,1]}] = [\Gamma(1)] = [\Gamma(0)] = [\gamma_{[0,0]}] = [c_{b_0}].$$

In other words, γ is null. Since q is a covering, this implies that Γ is null as well. ∎

We have shown that if a space is **semilocally simply-connected**, then it has a universal cover. So to provide an example of a space without a universal cover, it suffices to give an example of a space with a point which has no relatively simply connected neighborhood.

Example 2.26 (The Hawaiian earring)**.** Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0)$. So each such circle is tangent to the y-axis at the origin. Let $C = \cup_n C_n$. We claim that the origin has no relatively simply connected neighborhood. Indeed, let U be any neighborhood of the origin. Then for large enough n, the circle C_n is contained in U. A loop α that goes once around the circle C_n is not contractible in C. To see this, note that the map $r_n : C \longrightarrow S^1$ which collapses every circle except for C_n is a retraction. The loop $r \circ \alpha$ is not null, so α can't be null.

This example looks like an infinite wedge of circles, but it is not just a wedge. For instance, in each C_n consider an open interval U_n of radian length $1/n$ centered at the origin (or the open left semicircle, if you prefer). The union $U = \cup_n U_n$ of the U_n's is open in the infinite wedge of circles but not in C, since no ϵ-neighborhood of the origin is contained in U.

Wed, Feb. 21

3. The van Kampen Theorem

The focus of the next unit of the course will be on computation of fundamental groups.

One example we have already studied is the fundamental group of $S^1 \vee S^1$. We saw that this is the free group on two generators. We will see similarly that the fundamental group of $S^1 \vee S^1 \vee S^1$ is a free group on three generators. We will also want to compute the fundamental group of the two-holed torus (genus two surface), the Klein bottle, and more.

The main idea will be to decompose a space X into smaller pieces whose fundamental groups are easier to understand. For instance, if $X = U \cup V$ and we understand $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$, we might hope to recover $\pi_1(X)$.

Proposition 3.1. *Suppose that $X = U \cup V$, were U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then the smallest subgroup of $\pi_1(X)$ containing the images of both $\pi_1(U)$ and $\pi_1(V)$ is $\pi_1(X)$ itself.*

In group theory, we would say $\pi_1(X) = \pi_1(U)\pi_1(V)$.

Note that we really do need the assumption that $U \cap V$ is path-connected. If we consider U and V to be open arcs that together cover S^1, then both U and V are simply-connected, but their intersection is not path-connected. Note that here that the product of two trivial subgroups is not $\pi_1(S^1) \cong \mathbb{Z}$!

Proof. Let $\gamma : I \longrightarrow X$ be a loop at x_0. By the Lebesgue number lemma, we can subdivide the interval I into smaller intervals $[s_i, s_{i+1}]$ such that each subinterval is taken by γ into either U or V. We write γ_1 for the restriction of γ to the first subinterval. Suppose, for the sake of argument, that γ_1 is a path in U and that γ_2 is a path in V. Since $U \cap V$ is path-connected, there is a path δ_1 from $\gamma_1(1)$ to x_0. We may do this for each γ_i. Then we have

$$[\gamma] = [\gamma_1] * [\gamma_2] * [\gamma_3] * \cdots * [\gamma_n] = [\gamma_1 * \delta_1] * [\delta_1^{-1} * \gamma_2 * \delta_2] * \cdots * [\delta_{n-1}^{-1} * \gamma_n]$$

This expresses the loop γ as a product of loops in U and loops in V. ■

This is a start, but it is not the most convenient formulation. In particular, if we would like to use this to calculate $\pi_1(X)$, then thinking of the product of $\pi_1(U)$ and $\pi_1(V)$ *inside of* $\pi_1(X)$ is not so helpful. Rather, we would like to express this in terms of some external group defined in terms of $\pi_1(U)$ and $\pi_1(V)$. We have homomorphisms

$$\pi_1(U) \longrightarrow \pi_1(X), \qquad \pi_1(V) \longrightarrow \pi_1(X),$$

and we would like to put these together to produce a map from some sort of product of $\pi_1(U)$ and $\pi_1(V)$ to $\pi_1(X)$. Could this be the direct product $\pi_1(U) \times \pi_1(V)$? No. Elements of $\pi_1(U)$ commute with elements of $\pi_1(V)$ in the product $\pi_1(U) \times \pi_1(V)$, so this would also be true in the image of any homomorphism $\pi_1(U) \times \pi_1(V) \longrightarrow \pi_1(X)$.

What we want instead is a group freely built out of $\pi_1(U)$ and $\pi_1(V)$. The answer is the **free product** $\pi_1(U) * \pi_1(V)$ of $\pi_1(U)$ and $\pi_1(V)$. Its elements are finite length words $g_1 g_2 g_3 g_4 \ldots g_n$, where each g_i is in either $\pi_1(U)$ or in $\pi_1(V)$. Really, we use the reduced words, where none of the g_i is allowed to be an identity element and where if $g_i \in \pi_1(U)$ then $g_{i+1} \in \pi_1(V)$.

Example 3.2. We have already seen an example of a free product. The free group F_2 is the free product $\mathbb{Z} * \mathbb{Z}$.

Example 3.3. Similarly, the free group F_3 on three letters is the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Example 3.4. Let C_2 be the cyclic group of order two. Then the free product $C_2 * C_2$ is an infinite group. If we denote the nonidentity elements of the two copies of C_2 by a and b, then elements of $C_2 * C_2$ look like a, ab, $ababa$, $ababababa$, $bababa$, etc.

Note that there is a homomorphism $C_2 * C_2 \longrightarrow C_2$ that sends both a and b to the nontrivial element. The kernel of this map is all words of even length. This is the (infinite) subgroup generated by the word ab (note that $ba = (ab)^{-1}$). In other words, $C_2 * C_2$ is an extension of C_2 by the infinite cyclic group $\mathbb{Z}$. Another way to say this is that $C_2 * C_2$ is a semidirect product of C_2 with $\mathbb{Z}$.

The free product has a universal property, which should remind you of the property of the disjoint union of spaces $X \amalg Y$. First, for any groups H and K, there are inclusion homomorphisms $H \longrightarrow H * K$ and $K \longrightarrow H * K$.

Proposition 3.5. *Suppose that G is any group with homomorphisms $\varphi_H : H \longrightarrow G$ and $\varphi_K : K \longrightarrow G$. Then there is a (unique) homomorphism $\Phi : H * K \longrightarrow G$ which restricts to the given homomorphisms from H and K.*

In other words, the free product is the coproduct in the world of groups.

So Proposition 3.1 can be restated as follows:

Proposition 3.6 (weak van Kampen)**.** *Suppose that $X = U \cup V$, where U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then the natural homomorphism*

$$\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$$

is surjective.

Fri, Feb. 23

Now that we have a surjective homomorphism to $\pi_1(X)$, the next step is to understand the kernel N. Indeed, then the First Isomorphism Theorem will tell us that $\pi_1(X) \cong \big(\pi_1(U) * \pi_1(V)\big)/N$. Here is one way to produce an element of the kernel. Consider a loop α in $U \cap V$. We can then consider its image $\alpha_U \in \pi_1(U)$ and $\alpha_V \in \pi_1(V)$. Certainly these map to the same element of $\pi_1(X)$, so $\alpha_U \alpha_V^{-1}$ is in the kernel.

Proposition 3.7. *With the same assumptions as above, the kernel K of $\pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$ is the normal subgroup N generated by elements of the form $\alpha_U \alpha_V^{-1}$.*

Recall that the normal subgroup generated by the elements $\alpha_U \alpha_V^{-1}$ can be characterized either as (1) the intersection of all normal subgroups containing the $\alpha_U \alpha_V^{-1}$ or (2) the subgroup generated by all conjugates $g\alpha_U \alpha_V^{-1} g^{-1}$.

We will put off the proof of Propostion 3.7 for the moment. Assembling these recent results gives the van Kampen theorem:

Theorem 3.8 (Van Kampen)**.** *Suppose that $X = U \cup V$, where U and V are path-connected open subsets and both contain the basepoint x_0. If $U \cap V$ is also path-connected, then*

$$\pi_1(X, x_0) \cong \big(\pi_1(U, x_0) * \pi_1(V, x_0)\big)/N,$$

where $N \trianglelefteq \pi_1(U, x_0) * \pi_1(V, x_0)$ is the normal subgroup generated by elements of the form $\iota_U(\alpha)\iota_V(\alpha)^{-1}$, for $\alpha \in \pi_1(U \cap V, x_0)$.

There is another, more elegant, way to state the Van Kampen theorem.

Definition 3.9. Suppose given a pair of group homomorphisms $\varphi_G : H \longrightarrow G$ and $\varphi_K : H \longrightarrow K$. We define the **amalgamated free product** (or simply amalgamated product) to be the quotient

$$G *_H K = (G * K)/N,$$

where $N \trianglelefteq G * K$ is the normal subgroup generated by elements of the form $\varphi_G(h)\varphi_K(h)^{-1}$.

It is easy to check that *the amalgamated free product satisfies the universal property of the pushout in the category of groups.*

Theorem 3.10 (Van Kampen, restated)**.** *Let X be given as a union of two open, path-connected subsets U and V with path-connected intersection $U \cap V$. Then the inclusions of U and V into X induce an isomorphism*

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \xrightarrow{\cong} \pi_1(X).$$

Since the pasting lemma tells us that in this situation, X can itself be written as a pushout, the Van Kampen theorem can be interpreted as the statement that, under the given assumptions, the fundamental group construction takes a pushout of spaces to a pushout of groups.

One important special case of this result is when $U \cap V$ is simply connected.

Example 3.11. Take $X = S^1 \vee S^1$. Take U to be an open set containing one of the circles, plus an ϵ-ball around the basepoint in the other circle, and similarly for V with regard to the other circle. Then the intersection $U \cap V$ looks like an 'X' and is contractible, and U and V are both equivalent to S^1. We conclude from this that

$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} \cong F_2.$$

Example 3.12. Take $X = S^1 \vee S^1 \vee S^1$. We can take U to be a neighborhood of $S^1 \vee S^1$ and V to be a neighborhood of the remaining S^1. Then

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong (\mathbb{Z} * \mathbb{Z}) * \mathbb{Z} \cong F_3.$$

Example 3.13. Take $X = S^1 \vee S^2$. Take U to be a neighborhood of S^1 and V to be a neighborhood of S^2. We conclude from this that

$$\pi_1(S^1 \vee S^2) \cong \pi_1(S^1) * \pi_1(S^2) \cong \mathbb{Z}.$$

A natural question now is whether $\pi_1(X \vee Y)$ is always the free product of the $\pi_1(X)$ and $\pi_1(Y)$. Not quite, but a mild assumption allows us to make the conclusion. Note that in the $S^1 \vee S^1$ example, we needed to know that the neighborhoods U and V were homotopy equivalent to S^1 (and that the intersection was contractible).

Definition 3.14. We say that $x_0 \in X$ is a **nondegenerate basepoint** for X if x_0 has a neighborhood U such that x_0 is a deformation retract of U.

Proposition 3.15. *Let x_0 and y_0 be nondegenerate basepoints for X and Y, respectively. Then*

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

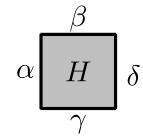
Proof. Suppose that x_0 is a deformation retract of the neighborhood $N_X \subseteq X$ and that y_0 is a deformation retract of the neighborhood $N_Y \subseteq Y$. Let $U = X \vee N_Y$ and $V = N_X \vee Y$. Then $U \cap V = N_X \vee N_Y$. The retracting homotopies for N_X and N_Y give $U \simeq X$, $V \simeq Y$, and $U \cap V \simeq *$. The van Kampen theorem then gives the conclusion. ■

Mon, Feb. 26

Lemma 3.16 (Square Lemma)**.** *Let α, β, γ, and δ be paths in X with*

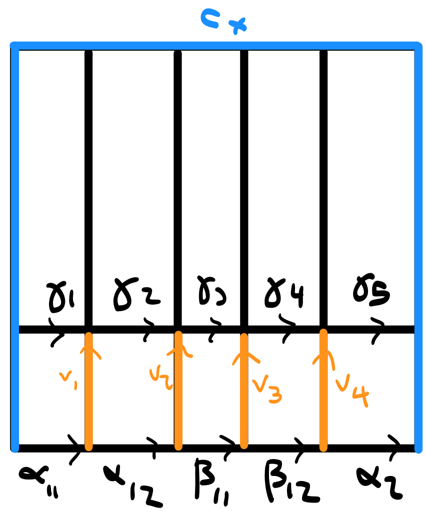
$$\alpha(0) = \gamma(0), \qquad \alpha(1) = \beta(0), \qquad \gamma(1) = \delta(0), \qquad \beta(1) = \delta(1).$$

Then path-homotopies $h : \alpha * \beta \simeq_p \gamma * \delta$ correspond bijectively to maps $H : I^2 \longrightarrow X$ as in the figure.



Proof of Proposition 3.7. Again, it is clear that the kernel K must contain the subgroup N. It remains to show that $K \leq N$. Consider an element of K. For simplicity, we assume it is $\alpha_1 \cdot \beta_1 \cdot \alpha_2$, where $\alpha_i \in \pi_1(U)$ and $\beta_1 \in \pi_1(V)$. The assumption that this is in K means that there exists a homotopy $H : I \times I \longrightarrow X$ from the path composition $\alpha_1 * \beta_1 * \alpha_2$ in X to the constant loop.

By the Lebesgue lemma, we may subdivide the square into smaller squares such that each small square is taken by H into either U or V. Again, we suppose for simplicity that this divides α_1 into α_{11} and α_{12} and β_1 into β_{11} and β_{12} (and α_2 is not subdivided).



Note that we cannot write

$$\alpha_1 \cdot \beta_1 \cdot \alpha_2 = \alpha_{11} \cdot \alpha_{12} \cdot \beta_{11} \cdot \beta_{12} \cdot \alpha_2$$

in $\pi_1(U) * \pi_1(V)$ since these are not all loops. But we can fix this, using the same technique as in the proof of Prop 3.1. In other words, we append a path δ back to x_0 at the end of every path on an edge of a square. If that path is in U (or V or $U \cap V$), we take δ in U (or V or $U \cap V$). Also, if the path already begins or ends at x_0, we do not append a δ. For convenience, we keep the same notation, but remember that we have really converted all of these paths to loops.

Let us turn our attention now to the homotopy H on the first (lower-left) square. Either H takes this into U or into V. If it is U, then we get a path homotopy in U $\alpha_{11} \simeq_p \gamma_1 \cdot v_1^{-1}$. If, on the other hand, H takes this into V, then it follows that α_{11} is really in $U \cap V$. This gives us a path homotopy in V $\alpha_{11} \simeq_p \gamma_1 \cdot v_1^{-1}$. But the group element α_{11} comes from $\pi_1(U)$ in the free product $\pi_1(U) * \pi_1(V)$. We would like to replace this with the element α_{11} from $\pi_1(V)$.

Lemma 3.17. *Let γ be any loop in $U \cap V$. Then, in the quotient group $Q = \big(\pi_1(U) * \pi_1(V)\big)/N$, the elements γ_U and γ_V are equivalent.*

Proof. The point is that

$$\gamma_V N = \gamma_U \gamma_U^{-1} \gamma_V N = \gamma_U \cdot \Big((\gamma^{-1})_U (\gamma^{-1})_V^{-1}\Big) N = \gamma_U N.$$

∎

From here on out, we work in the quotient group Q. The goal is to show that the original element $\alpha_1 \cdot \beta_1 \cdot \alpha_2$ is trivial in Q. According to the above, we can replace $(\alpha_1)_U (\beta_1)_V (\alpha_2)_U$ with either

$$(\gamma_1)_U (v_1^{-1})_U (\alpha_{12})_U (\beta_{11})_V (\beta_{12})_V (\alpha_2)_U$$

or

$$(\gamma_1)_V (v_1^{-1})_V (\alpha_{12})_U (\beta_{11})_V (\beta_{12})_V (\alpha_2)_U.$$

We then do the same with each of α_{12}, ..., α_2. The resulting expression will have adjacent terms v_i and v_i^{-1}. For the same i, these two loops may have the same label (U or V) or different labels. But by the lemma, we can always change the label if the loop lies in the intersection. So we get the path-composition of the paths along the top edges of the bottom squares. We then repeat the procedure, moving up rows until we get to the very top. But of course the top edges of the top squares are all constant loops. It follows that we end up with the trivial element (of Q). So $K = N$. ∎

The next application is the computation of the fundamental group of any *graph*. We start by specifying what we mean by a graph. Recall that $S^0 \subseteq \mathbb{R}$ is usually defined to be the set $S^0 = \{-1, 1\}$. For the moment, we take it to mean instead $S^0 = \{0, 1\}$ for convenience.

Definition 3.18. A **graph** is a 1-dimensional CW complex.

Of special importance will be the following type of graph.

Definition 3.19. A **tree** is a connected graph such that it is not possible to start at a vertex v_0, travel along successive edges, and arrive back at v_0 without using the same edge twice.

(Give examples and nonexamples)

Proposition 3.20. *Any tree is contractible. Even better, if v_0 is a vertex of the tree T, then v_0 is a deformation retract of T.*

Proof. We give the proof in the case of a finite tree. Use induction on the number of edges. If T has one edge, it is homeomorphic to I. Assume then that any tree with n edges deformation retracts onto any vertex and let T be a tree with $n+1$ edges. Let $v_0 \in T$. Now let $v_1 \in T$ be a vertex that is maximally far away from v_0 in terms of number of edges traversed. Then v_1 is the endpoint of a unique edge e, which we can deformation retract onto its other endpoint. The result is then a tree with n edges, which deformation retracts onto v_0. ∎

Wed, Feb. 28

Corollary 3.21. *Any tree is simply connected.*

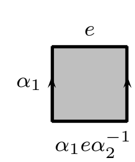
Definition 3.22. If X is a graph and $T \subseteq X$ is a tree, we say that T is a **maximal tree** if it is not contained in any other (larger) tree.

By Zorn's Lemma, any tree is contained in some maximal tree.

Theorem 3.23. *Let X be a connected graph and let $T \subseteq X$ be a maximal tree. The quotient space X/T is a wedge of circles, one for each edge not in the tree. The quotient map $q : X \longrightarrow X/T$ is a homotopy equivalence.*

Proof. Since T contains every vertex, all edges in the quotient become loops, or circles. To see that q is a homotopy equivalence, we first define a map $b : X/T \cong \bigvee S^1 \longrightarrow X$. Recall that to define a continuous map out of a wedge, it suffices to specify the map out of each wedge summand. Fix a vertex $v_0 \in T \subseteq X$. Pick a deformation retraction T down to v_0 as in Proposition 3.20. Then, for each vertex v, the homotopy provides a path $\alpha_v : v_0 \rightsquigarrow v$. Now suppose we have a circle corresponding to the edge e in X from v_1 to v_2. We then send our circle to the loop $\alpha_{v_1} e \alpha_{v_2}^{-1}$.

The composition $q \circ b$ on a wedge summand S^1 looks like $c * \mathrm{id} * c$ and is therefore homotopic to the identity. For the other composition, we want to extend the given homotopy on T to a homotopy on X. For simplicity, we give the argument in the case that $X = T \cup e$ has a single edge not in a maximal tree. We wish to define a homotopy $h : X \times I \longrightarrow X$, but we already have the homotopy on the subspace $T \times I$. It remains to specify the homotopy on $e \times I$, where we already have the homotopy on the edges $e_0 \times I$ and $e_1 \times I$. At time 0, the map $b \circ q$ takes e to $\alpha_1 * e * \alpha_2^{-1}$, whereas at time 1, the identity map takes e to e. We are now done by the Square Lemma (3.16).



∎

Corollary 3.24. *The fundamental group of any graph is a free group.*

We will use this to deduce an algebraic result about free groups. But first, a result about coverings of graphs.

Theorem 3.25. *Let $p : E \longrightarrow B$ be a covering, where B is a connected graph. Then E is also a connected graph.*

Proof. Recall our definition of a graph. It is a space obtained by glueing a set of edges to a set of vertices. Let B_0 be the vertices of B and B_1 be the set of edges. Let $E_0 \subseteq E$ be $p^{-1}(B_0)$ and define

$$E_1 \subseteq B_1 \times E_0$$

to be the set of pairs $(\alpha : \{0,1\} \longrightarrow B_0, v)$ such that $\alpha(0) = p(v)$. We then have compatible maps $E_0 \hookrightarrow E$ and $\amalg_{E_1} I \longrightarrow E$. The second map is given by the unique path-lifting property. These assemble to give a continuous map from the pushout

$$f : \hat{E} = E_0 \cup_{\coprod \partial I} \coprod_{e_1} I \longrightarrow E.$$

This pushout is a 1-dimensional CW complex, which is our definition of a graph.

Fri, Mar. 2

To see that f is surjective, let $x \in E$. Then $p(x)$ lies in some 1-cell β of B. Pick a path γ in B, lying entirely in β, from $p(x)$ to a vertex b_0. Then γ lifts uniquely to a path $\tilde{\gamma}$ starting at x in E. Write $v = \tilde{\gamma}(1)$. Then x lives in the 1-cell (β, v), so f is surjective.

The restriction of f to E_0 is injective, since this is the inclusion of the subset $E_0 \hookrightarrow E$. If y and z are two points of $\hat{E}$, where z is not a zero-cell and $f(y) = f(z)$, then $pf(y) = pf(z)$ in B. Since $pf(z)$ is not a 0-cell of B, we conclude that y is also not a 0-cell in $\hat{E}$. Now $pf(y)$ amd $pf(z)$ live in the same 1-cell of B, and since $f(y) = f(z)$ in E, uniqueness of lifts tells us that y and z live in the same 1-cell of $\hat{E}$. But the restriction of pf to the interior of this 1-cell is a homeomorphism onto the 1-cell of B. Since $pf(y) = pf(z)$, we conclude that $y = z$.

There are now several arguments for why this must be a homeomorphism. If B is a finite graph and E is a finite covering, we are done since E' is compact and E is Hausdorff (since B is Hausdorff). More generally, the map $\hat{E} \longrightarrow E$ is a map of covers which induces a bijection on fibers, so it must be an isomorphism of covers. ■

Now here is a purely algebraic statement, which we can prove by covering theory.

Theorem 3.26. *Any subgroup H of a free group G is free. If G is free on n generators and the index of H in G is k, then H is free on $1 - k + nk$ generators.*

Proof. Define B to be a wedge of circles, one circle for each generator of G. Then $\pi_1(B) \cong G$. Let $H \le G$ and let $p : E \longrightarrow B$ be a covering such that $p_*(\pi_1(E)) = H$. By the previous result, E is a graph and so $\pi_1(E)$ is a free group by the result from last time.

For the second statement, we introduce the **Euler characteristic** of a graph, which is defined as $\chi(B) = \#$ vertices $-\#$ edges. In this case, we have $\chi(B) = 1 - n$. Since H has index k in G, this means that G/H has cardinality k. But this is the fiber of $p : E \longrightarrow B$. So E has k vertices, and each edge of B lifts to k edges in E. So $\chi(E) = k - kn$.

On the other hand, we know from last time that E is homotopy equivalent to E/T, where $T \subseteq E$ is a maximal tree. Note that collapsing any edge in a tree does not change the Euler characteristic. The number of generators, say m of H, is then the number of edges in E/T, so we find that $\chi(E) = 1 - m$. Setting these equal gives

$$k - kn = 1 - m, \qquad \text{or} \qquad m = 1 - k + kn.$$

■

Mon, Mar. 5

We encountered an important idea in this proof, which can be defined more generally.

Definition 3.27. Let X be a CW complex having finitely many cells in each dimension (we saw that X is finite type). Then the **Euler characteristic** of X is

$$\chi(X) := \sum_{n\geq 0}(-1)^n \#\{n\text{-cells of } X\}.$$

In fact, the number $\chi(X)$ does not depend on the choice of CW structure on X, though this is far from obvious from the definition. We will see Euler characteristics again later in the course.

3.1. **The effect of attaching cells.** The van Kampen Theorem also gives an effective means of computing the fundamental group of a CW complex.

Given a space X and a map $\alpha : S^1 \longrightarrow X$, we may attach a disc along the map α to form a new space

$$X' = X \cup_\alpha D^2.$$

Since the inclusion of the boundary $S^1 \hookrightarrow D^2$ is null, it follows that the composition

$$\alpha : S^1 \longrightarrow X \longrightarrow X'$$

is also null. So we have effectively killed off the class $[\alpha] \in \pi_1(X)$.

We can use the van Kampen theorem to show that this is all that we have done.

Proposition 3.28. *Let X be path-connected and let $\alpha : S^1 \longrightarrow X$ be a loop in X, based at x_0. Write $X' = X \cup_\alpha D^2$. Then*

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X)/[\alpha].$$

Of course, we really mean the normal subgroup generated by α.

Proof. Consider the open subsets U and V of D^2, where $U = D^2 - \overline{B_{1/3}}$ and $V = B_{2/3}$. The map $\iota_{D^2} : D^2 \longrightarrow X'$ restricts to a homeomorphism (with open image) on the interior of D^2, so the image of V in X' is open and path-connected. Now let $U' = X \cup U$. Since this is the image under the quotient map $X \amalg D^2 \longrightarrow X'$ of the saturated open set $X \amalg U$, U' is open in X'. It is easy to see that U' is also path-connected.

Now U' and V cover X'. Since U deformation retracts onto the boundary, it follows that U' deformation retracts onto X. The open set V is contractible. Finally, the path-connected subset $U' \cap V$ deformation retracts onto the circle of radius $1/2$. Moreover, the map

$$\mathbb{Z} \cong \pi_1(U' \cap V) \longrightarrow \pi_1(U') \cong \pi_1(X)$$

sends the generator to $[\alpha]$. The van Kampen theorem then implies that

$$\pi_1(X') \cong \pi_1(X)/\langle\alpha\rangle.$$

■

Actually, we cheated a little bit in this proof, since in order to apply the van Kampen theorem, we needed to work with a basepoint in $U' \cap V$. A more careful proof would include the necessary change-of-basepoint discussion.

What about attaching higher-dimensional cells?

Proposition 3.29. *Let X be path-connected and let $\alpha : S^{n-1} \longrightarrow X$ be an attaching map for an n-cell in X, based at x_0. Write $X' = X \cup_\alpha D^n$. Then, if $n \geq 3$,*

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X).$$

Proof. The proof strategy is the same as for a 2-cell, so we don't reproduce it. The only change is that now $U' \cap V \simeq S^{n-1}$ is simply-connected. ■

Wed, Mar. 7

Example 3.30. If we attach a 2-cell to S^1 along the identity map $\mathrm{id} : S^1 \longrightarrow S^1$, we obtain D^2. We have killed all of the fundamental group. If we attach another 2-cell, we get S^2. Then $\chi(S^2) = 2 - 2 + 2 = 2$.

Attaching a 3-cell to S^2 via $\mathrm{id} : S^2 \longrightarrow S^2$ gives D^3. Attaching a second 3-cell gives S^3. The previous results tells us that all spaces obtained in this way (D^n and S^n) will be simply connected. Here we get $\chi(S^3) = 2 - 2 + 2 - 2 = 0$. More generally, we get

$$\chi(S^n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Example 3.31. ($\mathbb{RP}^n$) A more interesting example is attaching a 2-cell to S^1 along the double covering $\gamma_2 : S^1 \longrightarrow S^1$. Since this loop in S^1 corresponds to the element 2 in $\pi_1(S^1) \cong \mathbb{Z}$, the resulting space X' has $\pi_1(X') \cong \mathbb{Z}/2$. We have previously seen (last semester) that this is just the space $\mathbb{RP}^2$, since $\mathbb{RP}^2$ can be realized as the quotient of D^2 by the relation $x \sim -x$ on the boundary. This presents $\mathbb{RP}^2$ as a cell complex with a single 0-cell (vertex), a single 1-cell, and a single 2-cell. Then $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$.

We can next attach a 3-cell to $\mathbb{RP}^2$ along the double cover $S^2 \longrightarrow \mathbb{RP}^2$. The result is homeomorphic to $\mathbb{RP}^3$ by an analogous argument. By the above, this does not change the fundamental group, so that $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$, and we count $\chi(\mathbb{RP}^3) = 1 - 1 + 1 - 1 = 0$. In general, we have $\mathbb{RP}^n$ given as a cell complex with a single cell in each dimension. We have $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2$ for all $n \geq 2$, and

$$\chi(\mathbb{RP}^n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Example 3.32. ($\mathbb{CP}^n$) Recall that $\mathbb{CP}^1 \cong S^2$ is simply connected. Last semester, we showed that $\mathbb{CP}^n$ has a CW structure with a single cell in every even dimension. For example, $\mathbb{CP}^2$ is obtained from $\mathbb{CP}^1$ by attaching a 4-cell. It follows that every $\mathbb{CP}^n$ is simply-connected, and $\chi(\mathbb{CP}^n) = n+1$.

Let's look at a few more examples of CW complexes.

Example 3.33. (Torus) Attach a 2-cell to $S^1 \vee S^1$ along the map $S^1 \longrightarrow S^1 \vee S^1$ given by $aba^{-1}b^{-1}$, where a and b are the standard inclusions $S^1 \hookrightarrow S^1 \vee S^1$. We saw last semester that the resulting pushout is the torus, presented as a quotient of $D^2 \cong I^2$.

We claim that

$$\pi_1(T^2) \cong F_2/aba^{-1}b^{-1} \cong \mathbb{Z}^2.$$

Proposition 3.34. *The natural map $\varphi : F(a,b) \longrightarrow \mathbb{Z}^2$ defined by $\varphi(a) = (1,0)$ and $\varphi(b) = (0,1)$ induces an isomorphism*

$$F(a,b)/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2.$$

Proof. Let $K = \ker(\varphi)$ and let $N \trianglelefteq F(a,b)$ be the normal subgroup generated by $aba^{-1}b^{-1}$. By the First Isomorphism Theorem, $F(a,b)/K \cong \mathbb{Z}^2$, so it suffices to show that $N = K$. Since $aba^{-1}b^{-1} \in K$, it follows that $N \leq K$. Since $N \trianglelefteq K$, we wish to show that the quotient group K/N is trivial. Let $g = \overline{a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3}} \in K/N$. In K/N, we have $\overline{ab} = \overline{ba}$, so

$$\overline{a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3}} = \overline{a^{n_1+n_2+n_3}b^{k_1+k_2}}.$$

Since $g \in K$, we have $n_1 + n_2 + n_3 = 0$ and $k_1 + k_2 = 0$, so $\overline{g} = e$ in K/N. ■

So the answer coming from the van Kampen theorem matches our previous computation of $\pi_1(T^2)$.

In this cell structure on the torus, there is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell, so that

$$\chi(T^2) = 1 - 2 + 1 = 0.$$

Fri, Mar. 9

Example 3.35. (Klein bottle) One definition of the Klein bottle K is as the quotient of I^2 in which one opposite pair of edges is identified with a flip, while the other pair is identified without a flip. This leads to the computation

$$\pi_1(K) \cong F(a,b)/\langle aba^{-1}b\rangle.$$

For certain purposes, this is not the most convenient description. Cut the square along a diagonal and repaste the triangles along the previously flip-identified edges. The resulting square leads to the computation

$$\pi_1(K) \cong F(a,c)/\langle a^2c^2\rangle.$$

The equation $c = a^{-1}b$ allows you to go back and forth between these two descriptions.

Like the torus, the resulting cell complex has a single 0-cell, two 1-cells, and a single 2-cell, so

$$\chi(K) = 1 - 2 + 1 = 0.$$

The next example is not obtained by attaching a cell to $S^1 \vee S^1$.

Example 3.36. If we glue the boundary of I^2 according to the relation $abab$, the resulting space can be identified with $\mathbb{RP}^2$. Notice in this case that the four vertices do not all become identified. Rather they are identified in pairs, and we are left with two vertices after making the quotient. This example can be visualized by thinking of identifying the two halves of ∂D^2 via a twist. Using this cell structure, we get

$$\chi(\mathbb{RP}^2) = 2 - 2 + 1 = 1.$$

3.2. **The classification of surfaces.** These 2-dimensional cell complexes are all examples of **surfaces** (compact, connected 2-dimensional manifolds).

There is an important construction for surfaces called the **connected sum**.

Definition 3.37. Suppose M and N are surfaces. Pick subsets $D_M \subseteq M$ and $D_N \subseteq N$ that are homeomorphic to D^2 and remove their interiors from M and N. Write $M' = M - \mathrm{Int}(D_M)$ and $N' = N - \mathrm{Int}(D_N)$. Then the connected sum of M and N is defined to be

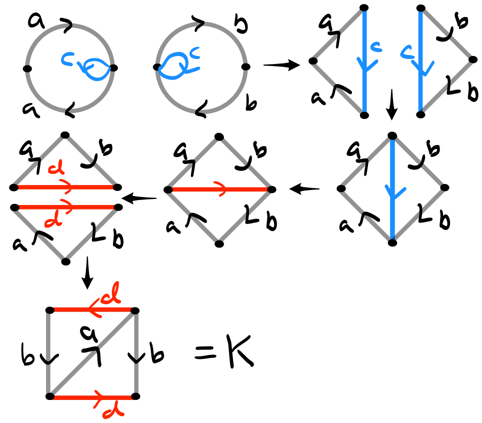
$$M\#N = M' \cup_{S^1} N',$$

where the maps $S^1 \longrightarrow M'$ and $S^1 \longrightarrow N'$ are the inclusions of the boundaries of the removed discs.

Example 3.38. If M is a surface, then the connect sum $M\#S^2$ is again homeomorphic to M.

Proposition 3.39. *The connected sum* $\mathbb{RP}^2 \# \mathbb{RP}^2$ *is homeomorphic to the Klein bottle,* K.

Proof. See the figure to the right. ■



Example 3.40. If M is a surface, then the connect sum $M \# T^2$ can be viewed as M with a "handle" glued on.

For example, consider $M = T^2$. Then $T^2 \# T^2$ looks liked a "two-holed torus". This is called M_2, the (orientable) surface of genus two. From the cell structure resulting from the picture, we see a wedge of four circles (let's call the generators of the circles a_1, b_1, a_2, b_2) with a two-cell attached along the element $[a_1, b_1][a_2, b_2]$. It follows that the fundamental group of M_2 is

$$F(a_1, b_1, a_2, b_2)/[a_1, b_1][a_2, b_2].$$

We also find that $\chi(M_2) = 1 - 4 + 1 = -2$.

Mon, Mar. 19

Example 3.41. (Surface of genus g) Similarly, if we take a connect sum of g tori, we get the surface of genus g, M_g. It has fundamental group

$$\pi_1(M_g) \cong F(a_1, b_1, \dots, a_g, b_g)/[a_1, b_1] \dots [a_g, b_g].$$

We now have $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$.

We are headed towards a "classification theorem" for compact surfaces, so let us now show that if $g_1 \neq g_2$ then M_{g_1} is not homeomorphic to M_{g_2}. We show this by showing they have different fundamental groups. As we have said already, understanding a group given by a list of generators and relations is not always easy, so we make life easier by considering the **abelianizations** of the fundamental groups.

The abelianization G_{ab} of G is the group defined by

$$G_{ab} = G/[G, G],$$

where $[G, G]$ is the (normal) subgroup generated by commutators.

Lemma 3.42. *The abelianization $F(a_1, \dots, a_n)_{ab}$ is the free abelian group $\mathbb{Z}^n$.*

Proof. We already did this in the case $n = 2$ for understanding the fundamental group of the torus, and the proof generalizes. ∎

The abelianization is characterized by a universal property. For a group G, let $q : G \longrightarrow G_{ab}$ be the quotient map. Then the universal property of the quotient gives the following result.

Proposition 3.43. *Let G be a group and A an abelian group. Then any homomorphism $\varphi : G \longrightarrow A$ factors uniquely as $G \xrightarrow{q} G_{ab} \xrightarrow{\overline{\varphi}} A$.*

When we apply this to the surface M_g, we get

Proposition 3.44. $\pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$.

Proof. Let $F = F(a_1, b_1, \dots, a_n, b_n)$, let $N \trianglelefteq F$ be the normal subgroup generated by (i.e. the normal closure of) the product $[a_1, b_1] \dots [a_g, b_g]$, and let $G = \pi_1(M_g) \cong F/N$. Since the quotient map $q : F \longrightarrow G$ is surjective, it follows that $q([F, F]) = [G, G]$. By the Third Isomorphism Theorem, we get

$$G_{ab} := G/[G, G] = G/q([F, F]) \cong F/[F, F] \cong \mathbb{Z}^{2g}.$$

∎

Lemma 3.45. *If $H \cong G$ then $H_{ab} \cong G_{ab}$.*

As a result, we see that if $g_1 \neq g_2$ then $\pi_1(M_{g_1}) \neq \pi_1(M_{g_2})$ because their abelianizations are not isomorphic.

Corollary 3.46. *If $g_1 \neq g_2$ them $M_{g_1} \ncong M_{g_2}$.*

Note that we have also distinguished all of these from S^2 (which has trivial fundamental group) and from $\mathbb{RP}^2$ (which has abelian fundamental group $\mathbb{Z}/2\mathbb{Z}$).

What about the Klein bottle K? We found before the break that $\pi_1(K) \cong F(a, b)/aba^{-1}b$. If we abelianize this fundamental group, we get

Proposition 3.47. *The abelianized fundamental group of the Klein bottle is*

$$\pi_1(K)_{ab} \cong (\mathbb{Z}\{a\} \times \mathbb{Z}\{b\})/(a + b - a + b) = \mathbb{Z}\{a\} \times \mathbb{Z}\{b\}/2b \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Proof. The idea is the same as in the previous examples. Under the quotient $F(a,b) \longrightarrow \mathbb{Z}\{a\} \times \mathbb{Z}\{b\}$, the element $aba^{-1}b$ is sent to $a+b-a+b$ (this is simply changing from multiplicative notation to additive notation. ∎

This group is different from all of the others, so K is not homeomorphic to any of the above surfaces. The last main example is

Example 3.48. ($\mathbb{RP}^2\#\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2$) Suppose we take a connect sum of g copies of $\mathbb{RP}^2$. We will call this surface N_g. Following the previous examples, we see that we get a fundamental group of

$$\pi_1(N_g) \cong F(a_1,\ldots,a_g)/a_1^2a_2^2\ldots a_g^2$$

and $\chi(N_g) = 1-g+1 = 2-g$. The abelianization is then

$$\pi_1(N_g)_{ab} \cong \mathbb{Z}^g/(2,2,\ldots,2).$$

Define a homomorphism $\varphi : \mathbb{Z}^g/(2,\ldots,2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{g-1}$ by

$$\varphi(n_1,\ldots,n_g) = (n_1, n_2-n_1, n_3-n_1,\ldots,n_g-n_1).$$

Then it is easily verified that φ is an isomorphism. In other words,

$$\pi_1(N_g)_{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}^{g-1}.$$

Ok, so we have argued that the compact surfaces S^2, M_g ($g \geq 1$), and N_g ($g \geq 1$) all have different fundamental groups and thus are not homeomorphic. The remarkable fact is that these are *all* of the compact (connected) surfaces.

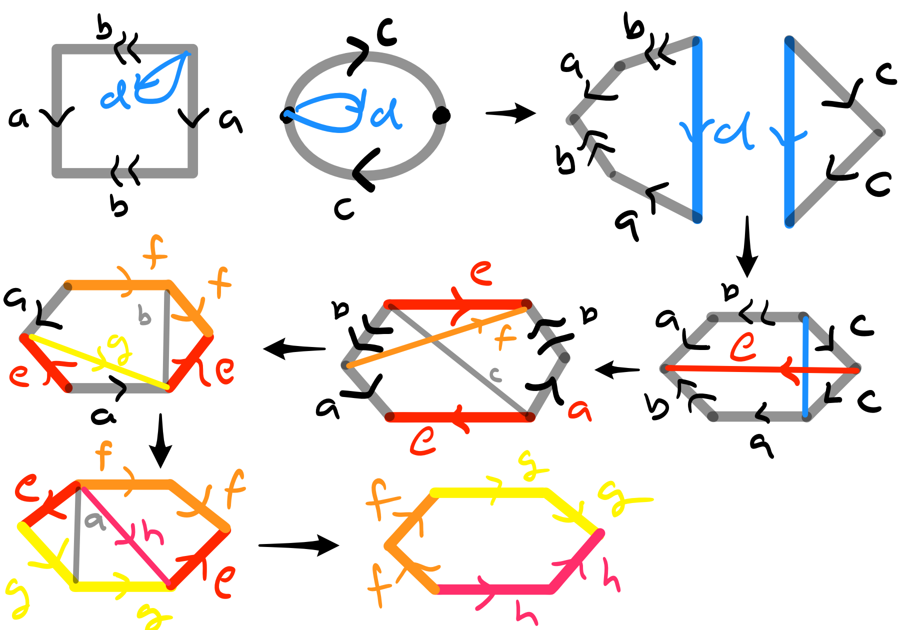
Theorem 3.49. *Every compact, connected surface is homeomorphic to some M_g, $g \geq 0$ or to some N_g, $g \geq 1$.*

Corollary 3.50. *If $\chi(M) = n$ is odd, then $M \cong N_{2-n}$*

All of these examples are formed by taking connected sums of T^2's or $\mathbb{RP}^2$'s. What happens if we mix them?

Lemma 3.51. $T^2\#\mathbb{RP}^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$.

In other words, one bad apple spoils the whole bunch. The proof is in the picture:



In particular, this implies that $M_g \# N_k \cong N_{2g+k}$.

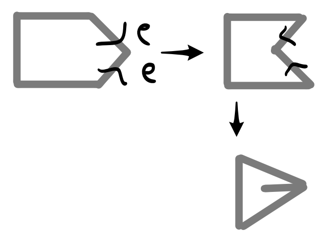
Proof of the theorem. Let M be a compact, connected surface. **We assume without proof** (see Prop 6.14 from Lee) that

- M is a 2-cell complex with a single 2-cell.
- the attaching map $\alpha : S^1 \longrightarrow M^1$ for the 2-cell has the following property: let U be the interior of a 1-cell. Then the restriction $\alpha : \alpha^{-1}(U) \longrightarrow U$ is a double cover. In other words, if we label ∂D^2 according to the edge identifications as we have done in the examples, each edge appears exactly twice. Note that this must happen since each interior point on the edge needs to have a half-disk on two sides.

So we can visualize M as a quotient of a $2n$-sided polygon.

As we said above, each edge appears exactly twice on the boundary of the two-cell. If the two occurrences have **opposite** orientations (as in the sphere), we say the pair is an **oriented pair**. If the two occurrences have the **same** orientation (as in $\mathbb{RP}^2$), we say this is a **twisted** pair. There will be 4 reductions in the proof!!

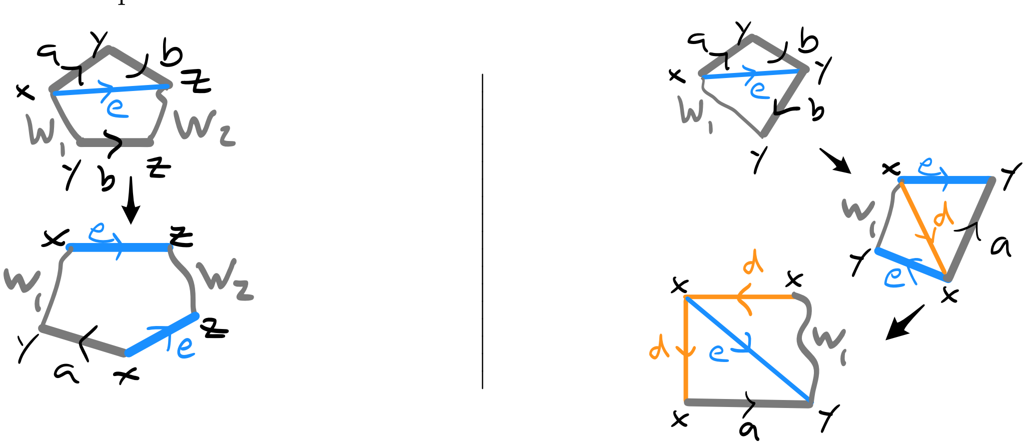
(1) If $M \cong S^2$, we are done, so suppose (for the rest of the proof) this is *not* the case. Then we can reduce to a cell structure with no *adjacent oriented pairs*. (Just fold these together.)



(2) We can reduce to a cell structure where all twisted pairs are adjacent.

If this creates any adjacent oriented pairs, fold them in.

(3) We can reduce to a cell structure with a single 0-cell. Suppose a is an edge from x to y and that $x \neq y$. Let b be the other edge connecting to y. By (1), b can't be a^{-1}. If $b = a$ then $x = y$. Suppose $b \neq a$, and write z for the other vertex on b. Then the edge b must occur somewhere else on the boundary. We use the moves in the pictures below, depending on whether the pair b is oriented or twisted.



This converts a vertex y into a vertex x. Note that this procedure does not separate any adjacent twisted pairs, since the adjacent twisted pair b gets replaced by d.

(4) Observe that any oriented pair a, a^{-1} is interlaced with another oriented pair b, b^{-1}. If not, we can write the boundary in the form $aW_1a^{-1}W_2$. Now, given our assumption and previous steps, no edge in W_1 gets identified with an edge in W_2. It follows that if the endpoints of a are x and y, then these two vertices never get identified with each other, as the vertex x cannot appear in W_1 and similarly y cannot appear in W_2.

(5) We can further arrange it so that there is no interference: the oriented pairs of edges occur as $aba^{-1}b^{-1}$ with no other edges in between. The proof is in the picture below, taken from p. 177 of Lee.

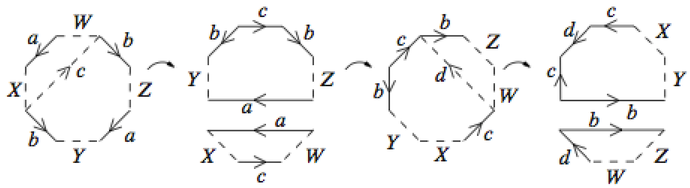


Fig. 6.22: Bringing intertwined complementary pairs together.

Now we are done by Lemma 3.51. M is homeomorphic either to a connect sum of projective planes or to a connect sum of tori. ■

Fri, Mar. 23

We saw in Corollary 3.50 that if $\chi(M)$ is odd, we can immediately identify the homeomorphism type of M. If $\chi(M)$ is even, this is not the case, as T^2 and K both have Euler characteristic equal to 0. To handle the even case, we make a definition.

Say that a surface M is **orientable** if it has a cell structure as above with no twisted pairs of edges.

Proposition 3.52. *A surface is orientable if and only if it is homeomorphic to some M_g.*

Proof. ($\Leftarrow$) Our standard cell structures for these surfaces have no twisted pairs of edges. ($\Rightarrow$) Apply the algorithm described in the above proof, starting with only oriented pairs of edges. Step 1 does not introduce any new edges. Step 2 can be skipped. Steps 3 cuts-and-pastes along a pair of oriented edges and so does not change the orientation of any edges. Step 4 does not change the surface. Step 5 again only cuts-and-pastes along oriented edges. It follows that in reducing to standard form, we do not introduce any twisted pairs of edges. ■

In fact, you should be able to convince yourself that a surface is orientable if and only if *every* cell structure as above has no twisted pairs. The point is that if you start with a cell structure involving some twisted pairs and you perform the reductions described in the proof, you will never get rid of any twisted pairs of edges.

The fact that the M_g can be embedded in $\mathbb{R}^3$ whereas the N_g cannot is precisely related to orientability. In general, you can embed a (smooth) n-dimensional manifold in $\mathbb{R}^{2n}$, but you can improve this to $\mathbb{R}^{2n-1}$ if the manifold is orientable. The definition we have given here depends on particular kinds of CW structures, but other definitions of orientability (in terms of homology) apply more widely.

In addition to the N_g's, the Möbius band is a 2-manifold that is famously non-orientable.

4. Higher homotopy groups

We have just been studying surfaces and have determined (well, at least given presentations for) their fundamental groups. We have also seen (on exam 1) that there are higher homotopy groups $\pi_n(X)$, so we might ask about the groups $\pi_n(M_g)$ and $\pi_n(N_k)$.

Recall, again from the exam, that any covering $E \longrightarrow B$ induces an isomorphism on all higher homotopy groups. So it suffices to understand the universal covers of these surfaces.

The first example would be $M_0 = S^2$, which is simply-connected. Note that this space is also the universal cover of $N_1 = \mathbb{RP}^2$, so these will have the same higher homotopy groups. We will come back to these on Monday.

Another example is the componentwise-exponential covering $q \times q : \mathbb{R}^2 \longrightarrow T^2$, which shows that T^2 has no higher homotopy groups. Note that we also could have deduced this using that

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$$

and that S^1 has no higher homotopy groups (also from Exam 1).

What about the Klein bottle K? Well, consider the relation on T^2 given by $(x, y) \sim (x + \frac{1}{2}, 1 - y)$. The quotient $T^2/\sim$ is K, and the quotient map $T^2 \longrightarrow K$ is a double cover. It follows that the universal cover of T^2, which is $\mathbb{R}^2$, is also the universal cover of K. So K also has no higher homotopy groups!

For the surfaces of higher genus, we start by generalizing the double cover $T^2 \longrightarrow K$.

Proposition 4.1. *If $g \geq 1$, then there is a double cover of N_g by M_{g-1}.*

Lemma 4.2. *Suppose that $p : E \longrightarrow B$ is a double cover of a surface B, and let W be another surface. Then there is a double cover $E\#W\#W \longrightarrow B\#W$.*

The lemma implies the proposition as follows:

Proof. We already know about the double cover $S^2 \longrightarrow \mathbb{RP}^2$, which is the case $g = 1$. Recall (Lemma 3.51) that $N_3 \cong \mathbb{RP}^2\#T^2$. By the lemma, we get a double cover $M_2 \cong S^2\#T^2\#T^2 \longrightarrow \mathbb{RP}^2\#T^2 \cong N_3$. By tacking on more copies of T^2, this handles the case of g odd.

We also discussed above the double cover $T^2 \longrightarrow K$, which is the case $g = 2$. By the lemma, we get a double cover $M_3 \cong T^2\#T^2\#T^2 \longrightarrow K\#T^2 \cong N_4$. By tacking on more copies of T^2, this handles the case of g even. ∎

Mon, Mar. 26

Last time, we saw that there is a double covering of the nonorientable surface N_g by the orientable surface M_{g-1}. It remains to find the universal cover of M_{g-1}.

Proposition 4.3. *For $g \geq 1$, the universal cover of M_g is $\mathbb{R}^2$.*

Sketch. We have already shown this for $g = 1$. In the higher genus case, this is more difficult. This is sometimes described using "hyperbolic" geometry. In that approach, $\mathbb{R}^2$ is replaced by the (homeomorphic) upper half-place, equipped with the hyperbolic metric. The idea is that you can tile the hyperbolic half-plane by polygons. Since M_g has a presentation as an (oriented) quotient of a polygon, this establishes a covering of M_g by the half-plane. ■

Ok, so we know that $\pi_n(\mathbb{RP}^2) \cong \pi_n(S^2)$. What are these groups? We will show later that $\pi_2(S^2) \cong \mathbb{Z}$. Just like for S^1, a generator for this group is the identity map $S^2 \longrightarrow S^2$. But the fascinating thing is that, in contrast to S^1, there are plenty of interesting higher homotopy groups! Here is a table of homotopy groups of spheres, taken from Wikipedia.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

There are several things to note in this table.

(1) We have $\pi_n(S^3) = \pi_n(S^2)$ for $n \geq 3$. There is a map $S^3 \longrightarrow S^2$ that induces this isomorphism on homotopy groups. It is the Hopf map η we studied before ($\mathbb{C}^2 - \{0\} \longrightarrow \mathbb{CP}^1$). This map is not a cover, since the fibers are circles. But this is a higher analogue of a covering: it is an S^1-bundle. The analogue of the "evenly covered neighborhoos" here is called "local triviality" of the bundle. This means that each point in $x \in \mathbb{CP}^1$ has a neighborhood U such that $\eta^{-1}(U) \cong S^1 \times U$. Remembering that a point in $\mathbb{CP}^1$ is of the form $x = [z_1 : z_2]$, consider the open sets $U_1 = \{[z_1 : z_2] | z_1 \neq 0\}$ and $U_2 = \{[z_1 : z_2] | z_2 \neq 0\}$. These certainly cover $\mathbb{CP}^1$, and the isomorphism

$$\eta^{-1}(U_1) \cong S^1 \times U_1$$

is

$$(z_1, z_2) \mapsto \left(\frac{z_1}{\|z_1\|}, [z_1 : z_2] \right).$$

A bundle still has a lifting property for paths and homotopies, but the lifts are no longer unique. This means that we can't necessarily lift an arbitrary map $Y \longrightarrow S^2$ up to a map $Y \longrightarrow S^3$, and it need not be true that *all* higher homotopy groups of S^2 are identified with those of S^3. It turns out that what happens here is that we have a "long exact sequence" relating the homotopy groups of S^3, S^2, and S^1 (most of which are trivial).

(2) We have $\pi_n(S^k) = 0$ if $n < k$. The argument is similar to the one that showed the higher spheres are all simply-connected. The main step is to show that any map $S^n \longrightarrow S^k$ is homotopic to a nonsurjective map if $n < k$.

(3) The answers are eventually constant on each diagonal. There is a suspension homomorphism $\pi_n(S^k) \longrightarrow \pi_{n+1}(S^{k+1})$ that induces these isomorphisms. The stable answer for $\pi_{k+n}(S^k)$ is known as the nth stable homotopy group of spheres and is written π_n^s. We have

$$\pi_0^s = \mathbb{Z}, \qquad \pi_1^s = \mathbb{Z}/2, \qquad \pi_2^s = \mathbb{Z}/2, \qquad \pi_3^s = \mathbb{Z}/24.$$

These groups are known out to around $n = 60$.

(4) Most of the unstable groups are finite. The only infinite ones are $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{4k-1}(S^{2k})$. The latter are all $\mathbb{Z}\times$(finite group). This is a theorem of J. P. Serre. This implies that all of the stable groups are finite, except $\pi_0^s = \mathbb{Z}$.

Ok, so homotopy groups are hard! But there are a few more examples of spaces whose homotopy groups are all known, so let's mention those before we abandon all hope and despair.

Example 4.4. Remember that we have a double cover $S^n \longrightarrow \mathbb{RP}^n$ inducing an isomorphism on all higher homotopy groups. But S^n does not have any homotopy groups until π_n, so this means that $\pi_k(\mathbb{RP}^n) = 0$ if $1 < k < n$. The inclusion $S^n \hookrightarrow S^{n+1}$, $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0)$ induces an inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$. As n gets higher, we lose more and more homotopy groups. In the limit, $S^\infty = \bigcup_n S^n$ has no homotopy groups (and in fact it is contractible). Similarly, $\mathbb{RP}^\infty$ has only a fundamental group of $\mathbb{Z}/2$ but no higher homotopy groups.

Example 4.5. There is an analogous story for $\mathbb{CP}^n$. Here, we have for every n, an S^1-bundle $S^{2n-1} \simeq \mathbb{C}^n - \{0\} \longrightarrow \mathbb{CP}^n$. This map induces an isomorphism on π_k for $k \geq 3$ and gives $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$. So the only nontrivial homotopy group of $\mathbb{CP}^\infty$ is $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$.

Wed, Mar. 28

Last time, we discussed higher homotopy groups of some familiar spaces. We saw that most of the M_g and N_g have no higher homotopy groups. On the other hand, basic spaces like S^2 and $\mathbb{RP}^2$ have very complicated (and unknown) higher homotopy groups. The other examples in which we had complete understanding of the higher homotopy groups were the infinite-dimensional complexes $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$. It turns out that this is quite typical: a finite cell complex almost always has infinitely many nontrivial homotopy groups!

5. Homology

This is rather disheartening. We think of a cell complex as an essentially finite amount of information. It would be nice if we only got finitely many algebraic objects out of it. There is such a construction: homology. As we will see, this will combine a number of the ideas we have recently encountered: the fundamental group and Euler characteristics. A good way to think about homology is that it is a more sophisticated version of the Euler characteristic.

We will deal with two versions of homology. The first, **singular homology**, is a good theoretical tool that is convenient for proving theorems. But it is not great for doing actual calculations. For that purpose, we will also consider **cellular homology**, which is defined for CW complexes. **Simplicial homology** is yet another version which is convenient for calculation, though we will not consider this version in our course.

5.1. **Singular homology.** Let Δ^n denote the standard n-simplex, which can be defined as

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, \quad t_i \geq 0\}.$$

We will denote by $v_i \in \Delta^n$ the vertex defined by $t_i = 1$ and $t_j = 0$ if $j \neq i$. Note that each "facet" of the simplex, in which we have restricted one of the coordinates to zero, is an $(n-1)$-dimensional simplex. More generally, if we set k of the coordinates equal to zero, we get a face which is an $(n-k)$-dimensional simplex.

Definition 5.1. Let X be a space. A **singular n-simplex** of X will simply be a continuous map $\Delta^n \longrightarrow X$.

Let $C_n^{\mathrm{Sing}}(X)$, or simply $C_n(X)$, be the free abelian group on the set of singular n-simplices of X. An element of $C_n(X)$ is referred to as a (singular) n-chain on X. Our goal is to assemble the $C_n(X)$, as n varies, into a "chain complex"

$$\dots \longrightarrow C_3(X) \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X).$$

To say that this is a chain complex just means that composing two successive maps in the sequence gives 0.

Fri, Mar. 30

EXAM DAY

Mon, Apr. 2

We wish to specify a homomorphism

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X).$$

Since $C_n(X)$ is a free abelian group, the homomorphism ∂_n is completely specified by its value on each generator, namely each n-simplex.

There are $n+1$ standard inclusions $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$, given by inserting 0 in position i in Δ^n.

Definition 5.2. The **singular boundary homomorphism**

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

is defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [\sigma \circ d^i].$$

Example 5.3.

(1) If σ is a 1-simplex (from v_0 to v_1), then

$$\partial_1(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] = [v_1] - [v_0].$$

(2) If σ is a 2-simplex with vertices v_0, v_1, and v_2, and edges e_{01}, e_{02}, and e_{12}, then

$$\partial_2(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] + [\sigma \circ d^2] = [e_{12}] - [e_{02}] + [e_{01}]$$

The claim is that this defines a chain complex. The signs have been inserted into the definition to make this work out.

Proposition 5.4. *The boundary squares to zero, in the sense that* $\partial_{n-1} \circ \partial_n = 0$.

Proof. We will use

Lemma 5.5. *For $i > j$, the composite*

$$\Delta^{n-2} \xrightarrow{d^j} \Delta^{n-1} \xrightarrow{d^i} \Delta^n \quad \textit{is equal to the composite} \quad \Delta^{n-2} \xrightarrow{d^{i-1}} \Delta^{n-1} \xrightarrow{d^j} \Delta^n.$$

Consider the case $i = 3$, $j = 1$, $n = 4$. We have

$$d^3(d^1(t_1, t_2, t_3)) = d^3(t_1, 0, t_2, t_3) = (t_1, 0, t_2, 0, t_3) = d^1(t_1, t_2, 0, t_3) = d^1(d^2(t_1, t_2, t_3)).$$

This argument generalizes.

For the proposition,

$$
\begin{aligned}
\partial_{n-1}\Big(\partial_n(\sigma)\Big) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i[\sigma\circ d^i]\right)\\
&= \sum_{i=0}^{n}(-1)^i\,\partial_{n-1}([\sigma\circ d^i])\\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j[\sigma\circ d^i\circ d^j]\\
&= \sum_{i=0}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j] + \sum_{i=0}^{n}\sum_{j\ge i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
\text{(changing bounds)}\quad &= \sum_{i=1}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j] + \sum_{i=0}^{n-1}\sum_{j\ge i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
\text{(Lemma)}\quad &= \sum_{i=1}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma\circ d^j\circ d^{i-1}] + \sum_{i=0}^{n-1}\sum_{j\ge i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
&= -\sum_{j=0}^{n-1}\sum_{i-1\ge j}(-1)^j(-1)^{i-1}[\sigma\circ d^i\circ d^j] + \sum_{i=0}^{n-1}\sum_{j\ge i}(-1)^i(-1)^j[\sigma\circ d^i\circ d^j]\\
&= 0.
\end{aligned}
$$

■

We have shown that any two successive simplicial boundary homomorphisms compose to zero, so that we have a chain complex. What do we do with a chain complex? Take homology!

Definition 5.6. If

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots$$

is a chain complex, then we define the **nth homology group** $\mathrm{H}_n(C_*, \partial_*)$ to be

$$\mathrm{H}_n(C_*, \partial_*) := \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Note that the fact that $\partial_n \circ \partial_{n+1} = 0$ implies that $\operatorname{im}\partial_{n+1}$ is a subgroup of $\ker\partial_n$, so that the definition makes sense. A complex (C_*, ∂_*) is said to be **exact** at C_n if we have equality $\ker\partial_n = \operatorname{im}\partial_{n+1}$. Thus the homology group $\mathrm{H}_n(C_*, \partial_*)$ "measures the failure of C_* to be exact at C_n."

Definition 5.7. Given a space X, we define the **singular homology groups** of X to be

$$\mathrm{H}_n(X;\mathbb{Z}) := \mathrm{H}_n(C_*(X), \partial_*).$$

Note that we only defined the groups $C_n(X)$ for $n \geq 0$. For some purposes, it is convenient to allow chain groups C_n for negative values of n, so we declare that $C_n(X) = 0$ for $n < 0$. This means that $\ker\partial_0 = C_0(X)$, so that $\mathrm{H}_0 = C_0(X)/\operatorname{im}\partial_1 = \operatorname{coker}(\partial_1)$.

Terminology: The group $\ker\partial_n$ is also known as the group of **n-cycles** and sometimes written Z_n. The group $\operatorname{im}(\partial_{n+1})$ is also known as the group of **boundaries** and sometimes written B_n.

Wed, Apr. 4

Remark 5.8. It is worth noting that since each $C_n(X)$ is free abelian and $\ker\partial_n$ and $\operatorname{im}\partial_{n+1}$ are both subgroups, they are necessarily also free abelian.

Example 5.9. Consider $X = *$. Then $C_n(\{*\}) = \mathbb{Z}\{\mathbf{Top}(\Delta^n, \{*\})\} \cong \mathbb{Z}$ for all n. The differential $\partial_n : C_n(\{*\}) \longrightarrow C_{n-1}(\{*\})$ takes the (constant) singular n-simplex c_n to the alternating sum

$$\sum_i (-1)^i c_{n-1} = \begin{cases} c_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

In other words, the chain complex is

$$\dots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

so that the only nonzero homology group is $\mathrm{H}_0(*) \cong \mathbb{Z}$.

But already for $X = \Delta^1$, the chain groups are infinite rank, and computing becomes impractical. On the other hand, the singular homology groups have good properties. For starters, we will discuss functoriality.

Given a map $f : X \longrightarrow Y$, we can compose any singular n-simplex of X with f to get a singular n-simplex of Y. This produces a function

$$f_n : \mathrm{Sing}_n(X) \longrightarrow \mathrm{Sing}_n(Y)$$

and therefore a homomorphism

$$f_n : C_n(X) \longrightarrow C_n(Y).$$

It remains to discuss how this interacts with homology.

Definition 5.10. Let (C_*, ∂_*^C) and (D_*, ∂_*^D) be chain complexes. Then a **chain map** $f_* : (C_*, \partial_*^C) \longrightarrow (D_*, \partial_*^D)$ is a sequence of homomorphisms $f_n : C_n \longrightarrow D_n$, for each n, such that each diagram

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ {\scriptstyle \partial_n^C}\downarrow & & \downarrow{\scriptstyle \partial_n^D} \\ C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} \end{array}$$

commutes for each n.

Since f_n is given by post-composition with f, whereas each term of ∂_n is given by precomposing with the face inclusions, it follows that the homomorphisms (f_*) on the singular chains assemble to produce a chain map.

We set up this definition in order to get

Proposition 5.11. *A chain map $f_* : (C_*, \partial_*^C) \longrightarrow (D_*, \partial_*^D)$ induces homomorphisms $f_n : \mathrm{H}_n(C_*, \partial_*^C) \longrightarrow \mathrm{H}_n(D_*, \partial_*^D)$ for each n.*

Proof. Let $x \in C_n$ be a cycle, meaning that $\partial^C(x) = 0$. Then $\partial^D(f_n(x)) = f_{n-1}(\partial^C(x)) = f_{n-1}(0) = 0$, so that $f_n(x)$ is a cycle in D_n. In order to get a well-defined map on homology, we need to show that if x is in the image of ∂_{n+1}^C, then $f_n(x)$ is in the image of ∂_{n+1}^D. But if $x = \partial_{n+1}^C(y)$, then $f_n(x) = f_n(\partial_{n+1}^C(y)) = \partial_{n+1}^D f_{n+1}(y)$, which shows that $f_n(x)$ is a boundary. ■

There is an obvious way to compose chain maps, so that chain complexes and chain maps form a category $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$.

Proposition 5.12. *The assignment $X \mapsto (C_*(X), \partial_*)$ and $f \mapsto f_*$ defines a functor*

$$C_* : \mathbf{Top} \longrightarrow \mathbf{Ch}_{\geq 0}(\mathbb{Z}).$$

Given the above discussion, it only remains to show that this construction takes identity morphisms to identity morphisms and that it preserves composition. We leave this as an exercise.

Note that the sequence of homology groups $\mathrm{H}_n(C_*, \partial_*^C)$ of a chain complex is not quite a chain complex, since there are no differentials between the homology groups. You can think of this as a degenerate case of a chain complex, in which all differentials are zero. But it is more common to simply call this a **graded abelian group**. If X_* and Y_* are graded abelian groups, then a graded map $f_* : X_* \longrightarrow Y_*$ is simply a collection of homomorphisms $f_n : X_n \longrightarrow Y_n$. Graded maps compose in the obvious way, so that we get a category **GrAb** of graded abelian groups. Then Proposition 5.11 is the main step in proving

Proposition 5.13. *Homology defines a functor*

$$\mathrm{H}_* : \mathbf{Ch}_{\geq \mathbf{0}}(\mathbb{Z}) \longrightarrow \mathbf{GrAb}.$$

The composition of two functors is always a functor. Thus Proposition 5.12 and Proposition 5.13 combine to yield

Proposition 5.14. *Singular homology defines a functor*

$$\mathrm{H}_*^{\mathrm{Sing}} : \mathbf{Top} \longrightarrow \mathbf{GrAb}.$$

This implies, for instance, that homeomorphic spaces have isomorphic singular homology groups.

Fri, Apr. 6

Last time, we discussed how a map of spaces induces a map on homology. Previously, we saw that the induced map on fundamental groups only depended on the homotopy class of the map, and we might ask the same question here.

Proposition 5.15. *Suppose that $f \simeq g$ as maps $X \longrightarrow Y$. Then f and g induce the same map on homology.*

Corollary 5.16. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then f induces an isomorphism on homology.*

Sketch of Proposition 5.15. See Theorem 13.8 of Lee for complete details.

If we have maps $f, g : X \longrightarrow Y$, it would be enough to show that their difference $f_* - g_*$ at the level of chains always takes values in the group of boundaries. Unfortunately, this is not always true, but it turns out to be true on cycles, which is enough to deduce the proposition. For simplicity, we consider the "universal" case, in which $Y = X \times I$ and f and g are the inclusions at time 0 and 1, respectively.

The idea is to define a homomorphism (called a "chain-homotopy") $h_n : C_n(X) \longrightarrow C_{n+1}(X \times I)$ for all n, satisfying the equation

$$h \circ \partial + \partial \circ h = g_* - f_*.$$

If you plug in a cycle x to this formula, you learn that $g_*(x) - f_*(x)$ is a boundary, so that f_* and g_* agree at the level of homology.

When $n = 0$, we simply take $h_0(x)$ to be the constand path in $X \times I$ from $(x, 0)$ to $(x, 1)$. At level 1, if σ is a path in X, we wish to define $h_1(\sigma) \in C_2(X \times I)$ with

$$h_0(\sigma(1) - \sigma(0)) + \partial \circ h_1(\sigma) = \sigma \times \{1\} - \sigma \times \{0\}.$$

We take $h_1(\sigma)$ to be the formal difference of simplices with vertices $(\sigma_0, 0)$, $(\sigma_1, 0)$, and $(\sigma_1, 1)$ and $(\sigma_0, 0)$, $(\sigma_0, 1)$, $(\sigma_1, 1)$. Similar formulas work in higher dimensions. ■

Example 5.17. We saw that the one-point space has homology groups nonvanishing only in dimension zero, given by the group $\mathbb{Z}$. It follows that the same is true for any contractible space, such as I^n or D^n or $\mathbb{R}^n$.

Mon, Apr. 9

5.2. **The functor** $\mathrm{H}_0(-)$**.**

Proposition 5.18. *If X is path-connected and nonempty, then* $\mathrm{H}_0(X) \cong \mathbb{Z}$*.*

Proof. Define $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ by sending each 0-simplex (i.e. point of X) to 1. As X is nonempty, the map ε is surjective. We claim that $\ker(\varepsilon) = B_0 = \mathrm{im}(\partial_1)$.

For any 1-simplex σ, $\partial_1(\sigma) = \sigma(1) - \sigma(0)$, so $\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$. This shows that $B_0 \subseteq \ker(\varepsilon)$.

Now suppose that $c = \sum_{i=1}^k n_i x_i$ is a 0-chain. Pick a point $x_0 \in X$, and, for each $i = 1, \dots, k$, pick a path $\alpha_i : x_0 \rightsquigarrow x_i$. Then $\partial_1(\alpha_i) = x_i - x_0$, so that $x_i \equiv x_0$ in $C_0(X)/B_0(X)$. Therefore $c \equiv (\sum_i n_i)x_0$ in $C_0(X)/B_0$. Now if $c \in \ker(\varepsilon)$, this means that $\sum_i n_i = 0$, so that $c \equiv 0$ in $C_0(X)/B_0$. In other words, $c \in B_0$. ■

To describe H_0 for a general space, we first discuss how path components interact with homology.

Proposition 5.19. *Let $\{X_\alpha\}$ be the set of path-components of X and $\iota_\alpha : X_\alpha \longrightarrow X$ the inclusions. These induce an isomorphism*

$$\bigoplus_\alpha \mathrm{H}_*(X_\alpha) \cong \mathrm{H}_*(X).$$

Proof. Since the image of any singular n-simplex must be contained in a single path-component, we get already a splitting of the chain complexes

$$\bigoplus_\alpha C_*(X_\alpha) \cong C_*(X).$$

This produces the splitting on the level of homology. ■

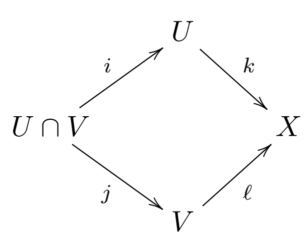
Corollary 5.20. *For any space X, $\mathrm{H}_0(X)$ is free abelian on the set of path-components of X. In other words,*

$$\mathrm{H}_0(X) \cong \mathbb{Z}\{\pi_0(X)\}.$$

5.3. **The Mayer-Vietoris Sequence.** One of the fundamental tools for computing homology is the Mayer-Vietoris sequence, which is analogous to the van Kampen theorem for the fundamental group. First, some terminology.

Recall (from just before Definition 5.7) that we say that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if it has no homology, meaning that $\mathrm{im}(f) = \ker(g)$. Very often, we encounter an exact sequence in which either A or C is 0. If $A = 0$, then the image of f must also be zero, so that g must be injective. Similarly, if $C = 0$, then the kernel of g must be all of B, so that f must be surjective. For a longer sequence, such as $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow \dots$, we say it is exact if it is so at each group in the sequence.

We consider a space X with open subsets U and V. We will denote the inclusions as in the diagram



Theorem 5.21 (Mayer-Vietoris long exact sequence)**.** *Let X be a space, and let U and V be open subsets with $U \cup V = X$. Then there is a long exact sequence in homology*

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(U \cap V) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(U) \oplus \mathrm{H}_n(V) \xrightarrow{k_* - \ell_*} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(U \cap V) \xrightarrow{i_* \oplus j_*} \dots$$

Before proving the theorem, we give a sample application.

Example 5.22. ($\mathrm{H}_*(S^k)$) Combining Example 5.9 with Proposition 5.19 gives that

$$\mathrm{H}_i(S^0) \cong \begin{cases} \mathbb{Z}^2 & i = 0 \\ 0 & \text{else.} \end{cases}.$$

Wed, Apr. 11

We use the Mayer-Vietoris sequence to compute the homology of the higher spheres. We argue by induction that for $k > 0$,

$$\mathrm{H}_i(S^k) \cong \begin{cases} \mathbb{Z} & i = 0, k \\ 0 & \text{else.} \end{cases}.$$

The base case is S^1. Take U and V to be the open subsets of S^1 given by removing the north and south poles, respectively. Notice that U and V are both contractible and that $U \cap V$ deformation retracts to the equatorial S^0. Thus the Mayer-Vietoris sequence becomes

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(S^0) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(*) \oplus \mathrm{H}_n(*) \xrightarrow{k_* - \ell_*} \mathrm{H}_n(S^1) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(S^0) \xrightarrow{i_* \oplus j_*} \dots.$$

Note that when n is larger than 1, then $\mathrm{H}^n(S^1)$ is flanked by two zero groups and must therefore by zero. We are left then only with the exact sequence

$$0 \longrightarrow \mathrm{H}_1(S^1) \xrightarrow{\partial_1} \mathrm{H}_0(S^0) \xrightarrow{i_* \oplus j_*} \mathrm{H}_0(*) \oplus \mathrm{H}_0(*) \xrightarrow{k_* - \ell_*} \mathrm{H}_0(S^1) \cong \mathbb{Z} \longrightarrow 0.$$

This becomes

$$0 \longrightarrow \mathrm{H}_1(S^1) \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{(1\ -1)} \mathbb{Z} \longrightarrow 0.$$

It follows that the image of ∂_1 is the subgroup generated by $(1, -1)$, so that $\mathrm{H}_1(S^1) \cong \mathbb{Z}$.

Now for the induction step, suppose the formula holds for $\mathrm{H}_*(S^k)$ and consider S^{k+1}. We again take U and V to be the complements of the poles in S^{k+1}. Now the Mayer-Vietoris sequence becomes

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(S^k) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(*) \oplus \mathrm{H}_n(*) \xrightarrow{k_* - \ell_*} \mathrm{H}_n(S^{k+1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(S^k) \xrightarrow{i_* \oplus j_*} \dots.$$

We know by Proposition 5.18 that $\mathrm{H}_0(S^{k+1}) \cong \mathbb{Z}$, and the exact sequence gives that $\mathrm{H}_{n+1}(S^{k+1}) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(S^k)$ is an isomorphism for $n \geq 1$. Finally, the group $\mathrm{H}_1(S^{k+1})$ is in the exact sequence

$$0 \longrightarrow \mathrm{H}_1(S^{k+1}) \xrightarrow{\partial_1} \mathrm{H}_0(S^k) \xrightarrow{i_* \oplus j_*} \mathrm{H}_0(*) \oplus \mathrm{H}_0(*) \longrightarrow \mathrm{H}_0.$$

The map $i_* \oplus j_*$ is the diagonal map $\mathbb{Z} \longrightarrow \mathbb{Z}^2$, which is injective. It follows that $\mathrm{H}_1(S^{k+1}) = 0$.

Fri, Apr. 13

The main step in the proof of the Mayer-Vietoris theorem is the following result. We say that a sequence $0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{q} C_* \longrightarrow 0$ of chain complexes is exact if each sequence $0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{q} C_n \longrightarrow 0$ is exact.

Proposition 5.23. *A short exact sequence* $0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{q} C_* \longrightarrow 0$ *of chain complexes induces a long exact sequence in homology*

$$\ldots \longrightarrow \mathrm{H}_{n+1}(C) \xrightarrow{\delta} \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{q_*} \mathrm{H}_n(C) \xrightarrow{\delta} \mathrm{H}_{n-1}(A) \longrightarrow \ldots$$

Proof. We start with the construction of the "connecting homomorphism δ". Thus let $c \in C_n$ be a cycle. Choose a lift $b \in B_n$, meaning that $q(b) = c$. We then have $q(\partial_n(b)) = \partial_n(q(b)) = \partial_n(c) = 0$. Since the rows are exact, we have $\partial_n(b) = i(a)$ for some unique $a \in A_{n-1}$, and we define

$$\delta(c) := a.$$

$$\begin{array}{ccccc} & & b & \longmapsto & c \\ & & \downarrow & & \downarrow \\ a & \longmapsto & \partial(b) & \longmapsto & 0 \end{array}$$

It remains to see how a depends on the choice of b. Thus let $d \in \ker(q)$, so that $q(b+d) = c$. By exactness, we have $d = i(e)$ for some $e \in A_n$. Then

$$i(a + \partial_n(e)) = \partial_n(b) + i(\partial_n(e)) = \partial_n(b) + \partial_n(i(e)) = \partial_n(b) + \partial_n(d) = \partial_n(b+d),$$

so that $\delta(c) = a + \partial_n(e) \sim a$. In other words, a specifies a well-defined homology class.

Since we want δ to be well-defined not only on cycles but also on homology, we need to show that if c is a boundary, then $\delta(c) \sim 0$. Thus suppose $c = \partial(c')$. We can then choose b' such that $q(b') = c'$. It follows that $\partial(b')$ would be a suitable choice for b. But then $\partial(b) = \partial(\partial(b')) = 0$, so that $\delta(c) = 0$.

Exactness at B: First, we see that $q_* \circ i_* = 0$ since this is already true at the chain level. Now suppose that $b \in \ker(q_*)$. This means that $q(b) = \partial(c)$ for some $c \in C_{n+1}$. Now choose a lift $d \in B_{n+1}$ of c. Then we know

$$q(\partial(d)) = \partial(q(d)) = \partial(c) = q(b).$$

In other words, $q(b - \partial(d)) = 0$, so that we must have $b - \partial(d) = i(a)$ for some a. Since $b \sim b - \partial(d)$, we are done.

Exactness at C: We first show that $\delta \circ q_* = 0$. Thus let $b \in B_n$ be a cycle. We wish to show that $\delta(q_*(b)) = 0$. But the first step in constructing $\delta(q(b))$ is to choose a lift for $q(b)$, which we can of course take to be b. Then $\partial(b) = 0$, so that $a = 0$ as well.

Now suppose that $c \in C_n$ is a cycle that lives in the kernel of δ. This means that $a = \partial(e)$ for some e. But then $b - i(e)$ is a cycle, and $q(b - i(e)) = c$, so c is in the image of q_*.

Exactness at A: First, we show that $i_* \circ \delta = 0$. Let $c \in C_n$ be a cycle. Then if $\delta(c) = a$, then by construction, we have $i(a) = \partial(b) \sim 0$, so that $i_* \circ \delta = 0$.

Finally, suppose that $a \in A_n$ is a cycle that lives in $\ker i_*$. Then $i(a) = \partial(b)$ for some b, but then $a = \delta(q(b))$. ∎

Sketch of Theorem 5.21. We would like to apply Proposition 5.23 to the sequence

$$0 \longrightarrow C_*(U \cap V) \xrightarrow{i_* + j_*} C_*(U) \oplus C_*(V) \xrightarrow{k_* - \ell_*} C_*(X) \longrightarrow 0.$$

The problem is that this is not exact at $C_*(X)$. The reason is that not every singular n-simplex in X is contained entirely in U or V. Instead, we introduce the subcomplex $C_*^{U,V}(X)$, where $C_n^{U,V}(X)$ is the free abelian group on simplices which are entirely contained in either U or V.

We claim that the inclusion $C_*^{U,V}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence. We need to define a homotopy inverse $f : C_*(X) \longrightarrow C_*^{U,V}(X)$. The idea is to use "barycentric subdivision". The subdivision of an n-simplex expresses it as the union of smaller n-simplices. By the Lebesgue Number Lemma, repeated barycentric subdivision will eventually decompose any singular n-simplex of X into a collection of n-simplices, each of which is either contained in A or in B. This subdivision allows you to define a chain map f. You then show that subdivision of simplices is chain-homotopic to the identity. See Proposition 2.21 of Hatcher for a much more detailed discussion. ■

Mon, Apr. 16

5.4. **The Hurewicz Theorem.** We saw previously that $\mathrm{H}_0(X) \cong \mathbb{Z}\{\pi_0(X)\}$. What about $\mathrm{H}_1(X)$? It turns out this is closely related to $\pi_1(X)$. First note that given a map $\alpha : S^1 \longrightarrow X$, we get an induced map $\mathbb{Z} \cong \mathrm{H}_1(S^1) \longrightarrow \mathrm{H}_1(X)$. If we pick a preferred generator for $\mathrm{H}_1(S^1)$, for example the 1-simplex $\Delta^1 \longrightarrow S^1$ which is the quotient map

$$\Delta^1 \cong I \longrightarrow I/\partial I \cong S^1,$$

then this picks out a particular element of $\mathrm{H}_1(X)$.

Proposition 5.24. *This element $\alpha_*(1) \in \mathrm{H}_1(X)$ only depends on the homotopy class of α.*

Proof. This follows from Proposition 5.15. ∎

We then define the **Hurewicz** function

$$h : \pi_1(X, x_0) \longrightarrow \mathrm{H}_1(X)$$

by $h([\alpha]) = \alpha_*(1)$. By the proposition, this is well-defined on homotopy-classes.

Theorem 5.25 (Hurewicz)**.** *Assume that X is path-connected. Then h induces an isomorphism*

$$H_1(X) \cong \pi_1(X, x_0)_{ab}.$$

Proof. We first show that h is a group homomorphism. First, it preserves identity elements since if we consider the constant loop at x_0 as a 1-cycle, we can express it as the boundary of the constant 2-simplex at x_0. Next, suppose we have two loops α and β. We wish to show that $h(\alpha \cdot \beta) = h(\alpha) + h(\beta)$. Either by using the Square Lemma (Lemma 3.16) or by writing one down explicitly, we can define a 2-simplex $\sigma_{\alpha,\beta}$ whose restriction to the boundary is the three edges α, $\alpha\beta$, and β. Then $\partial(\sigma_{\alpha,\beta}) = \alpha - \alpha \cdot \beta + \beta$. This shows that $h(\alpha \cdot \beta) = h(\alpha) + h(\beta)$.

Since we now know that h is a homomorphism, we can use the universal property of abelianization to factor

$$h : \pi_1(X) \longrightarrow \mathrm{H}_1(X)$$

through $\hat{h} : \pi_1(X)_{ab} \longrightarrow \mathrm{H}_1(X)$. It remains to show that $\hat{h}$ is bijective.

(Surjectivity): For each $x \in X$, pick a path $p_x : x_0 \rightsquigarrow x$. We also write $p : C_0(X) \longrightarrow C_1(X)$ for the resulting function. Now for each 1-simplex a, we can define a loop $\tilde{a}$ at x_0 by $p_{a(0)} \cdot a \cdot \overline{p_{a(1)}}$. Then

$$h(\tilde{a}) = [p_{a(0)}] + [a] + [\overline{p_{a(1)}}] = [a] + [p \circ \partial(a)].$$

Now take an arbitrary 1-cycle $c = \sum_i n_i a_i$. Then we get

$$h(\tilde{a_1}^{n_1} \tilde{a_2}^{n_2} \cdots \tilde{a_k}^{n_k}) = \sum_i n_i[a_i] + n_i[p \circ \partial(a_i)] = [c] + p([\partial(c)]) = [c]$$

since c was assumed to be a cycle.

Wed, Apr. 18

(Injectivity): Let $\alpha \in \pi_1(X)$ be in the kernel of h. We wish to show that α is trivial in $\pi_1(X)_{ab}$. If $h(\alpha) = 0$, this means that α, when considered as a 1-simplex, is a boundary. Suppose, for example, that

$$\alpha = \partial(\sigma)$$

for some 2-simplex $\sigma : \Delta^2 \longrightarrow X$. But $\partial(\sigma) = \sigma_{1,2} - \sigma_{0,2} + \sigma_{0,1}$, so if this is equal to α in $C_1(X)$, then α must be either $\sigma_{0,1}$ or $\sigma_{1,2}$, and the other of these edges must agree with $\sigma_{0,2}$. Write β for the path $\sigma_{0,1}$. Then, by the square lemma, the two-simplex σ gives rise to a path-homotopy $\alpha\beta \simeq_p \beta$. In other words, $\alpha \simeq_p c_{x_0}$.

The trouble is that, in general, there is no reason to expect α to be the differential on a single 2-simplex. Rather, we expect to have

$$\alpha = \partial(\sum n_i \sigma_i).$$

Again, from the square lemma, each of these 2-simplices σ_i will give rise to a path-homotopy. All of the faces of the σ_i's cancel in $C_1(X)$, to leave only α. If we try to do the same manipulation in $\pi_1(X)$, using the path-homotopies, we need to allow ourselves to commute elements, since this can happen in $C_1(X)$ to allow for the cancellation there. So if we abelianize $\pi_1(X)$, we can perform the same cancellation to show that $[\alpha] = [c_{x_0}] \in \pi_1(X)_{ab}$. ■

Example 5.26. Recall from Proposition 3.44 that $\pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$. It follows that

$$\mathrm{H}_1(M_g) \cong \mathbb{Z}^{2g}.$$

Example 5.27. Recall from Example 3.48 that $\pi_1(N_g)_{ab} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows that

$$\mathrm{H}_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In fact, a stronger version of the Hurewicz theorem holds. We will not prove the stronger version.

Theorem 5.28. *Suppose that $\pi_k(X) = 0$ for $k < n$, where $n > 1$. Then $\mathrm{H}_n(X) \cong \pi_n(X)$.*

Corollary 5.29. *Let $n > 1$. Then* $\pi_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & 0 < k < n. \end{cases}$.

Proof. We already showed that this is the homology of the sphere. Since S^n is simply connected, Theorem 5.28 gives that $\pi_2(S^n) \cong \mathrm{H}_2(S^n)$. If $n = 2$, this is $\mathbb{Z}$ and we are done. If $n > 2$, this is 0, and then we apply the Hurewicz theorem at level 3. Repeat until you reach the first nonzero homology group. ■

5.5. **Cellular homology.** While singular homology is defined for all spaces and is nicely functorial, it is not so practical for computing by hand. For this purpose, we introduce cellular homology, which is defined for CW complexes.

Recall that at the end of last semester, we defined the **degree** of a map $f : S^1 \longrightarrow S^1$ by considering the induced map on fundamental groups. This map is multiplication by some integer, which we called the degree. If f was not based, the definition of degree involved the change-of-basepoint homomorphism. But now that we know about (singular) homology, there is a simpler definition, which works equally well in higher dimensions.

Definition 5.30. Let $f : S^n \longrightarrow S^n$ be any map. for $n \geq 1$. Then the induced map on homology

$$f_* : \mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n)$$

is multiplication by some integer d. We define the **degree** of f to be this integer d.

Fri, Apr. 20

Definition 5.31. Let X be a CW complex. Define the group $C_n^{cell}(X)$ of cellular n-chains by

$$C_n^{cell} := \mathbb{Z}\{n\text{-cells of } X\}.$$

To specify the differential $d_n : C_n(X) \longrightarrow C_{n-1}(X)$, we need to give the coefficients in

$$d_n(f) = \sum n_i e_i.$$

Here f is an n-cell, which is described by its attaching map $S^{n-1} \xrightarrow{f} \mathrm{sk}_{n-1}X$. The coefficient n_i in the expansion is the degree of the map

$$S^{n-1} \xrightarrow{f} X^{n-1} \twoheadrightarrow X^{n-1}/X^{n-2} \cong \bigvee S^{n-1} \xrightarrow{e_i} S^{n-1}.$$

This works well if $n-1 \geq 1$. The d_1 is defined similarly. A 1-cell e is determined by the attaching map, which simply specifies the endpoints $e(1)$ and $e(0)$. We define $d_1(e) = e(1) - e(0)$.

We now define the **cellular homology groups** to be the homology of this complex:

$$\mathrm{H}_n^{cell}(X) := \mathrm{H}_n(C_*^{cell}(X)).$$

On the face of it, this definition does not make sense, since we have not verified that $d \circ d = 0$. Probably the simplest way to establish this is to recognize that $C_n^{cell}(X) \cong \mathrm{H}_n(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X))$. Then the cellular differential can be viewed as the connecting homomorphism in a Mayer-Vietoris sequence. See the dicsussion above Theorem 2.35 of Hatcher for more details.

This definition of homology might sound complicated, but in practice it is quite simple. For instance, if our CW complex has a single 0-cell, then each 1-cell must be a loop, and the d_1-differential is just zero. Another immediate consequence of the definition is the following.

Proposition 5.32. *Suppose that X is an n-dimensional CW complex. Then $\mathrm{H}_k^{cell}(X) = 0$ for $k > n$.*

This is simply because X has no cells above dimension n, so that $C_k^{cell}(X) = 0$ if $k > n$.

Let's look at some examples.

Example 5.33. Take $X = S^2$. Pick the CW structure having a single vertex and a single 2-cell. Then $C_1(X) = 0$, so both d_2 and d_1 must be the zero map. The chain complex $C_*(S^2)$ is

$$\mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z}.$$

Here we get $H_0 = H_2 = \mathbb{Z}$ and $H_1 = 0$. The same would for any S^n, with $n \geq 2$.

Example 5.34. Take $X = S^2$. Pick the CW structure having a single vertex, a single edge, and two 2-cells attached via the identity map $S^1 \cong S^1$. Then $C_0(S^2) = C_1(S^2) = \mathbb{Z}$ and $C_2(S^2) = \mathbb{Z}^2$. The map

$$d_1 : C_1 = \mathbb{Z} \longrightarrow C_0 = \mathbb{Z}$$

is $d_1(e) = 0$ since the edge e is a loop. If we write f_1 and f_2 for the 2-cells, we see that $d_2(f_1) = d_2(f_2) = e$. Thus the resulting chain complex is

$$\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Here we see that $H_0 \cong \mathbb{Z}$ since $d_1 = 0$, so that $B_0 = 0$ and $H_0 = Z_0 = \mathbb{Z}$. Next, the statement $d_1 = 0$ also means that $Z_1 = C_1 = \mathbb{Z}$, and we see that d_2 is surjective, so that $B_1 = Z_1 = C_1$. It follows that $H_1 \cong \mathbb{Z}$. Finally, the kernel of d_2 is the cyclic subgroup of $\mathbb{Z}^2$ generated by $(1,-1)$, so $H_2 = Z_2 \cong \mathbb{Z}$.

Example 5.35. Take $X = S^2$. Pick the CW structure having two cells in each degree ≤ 2. Here each attaching map $S^{n-1} \longrightarrow X^{n-1}$ is an identity map. Write x_1 and x_2 for the vertices and e_1 and e_2 for the edges. We have $d_1(e_i) = x_2 - x_1$. Similarly, we have $d_2(f_i) = e_1 - e_2$. The resulting chain complex is

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2$$

Here, the differential d_1 has image the subgroup generated by $(-1,1)$, so $H_0 \cong \mathbb{Z}^2/(-1,1) \cong \mathbb{Z}$. The kernel of d_1 is the subgroup generated by $(1,-1)$, which is the image of d_2, so $H_1 = 0$. The kernel of d_2 is again the subgroup generated by $(-1,1)$, so that $H_2 \cong \mathbb{Z}$.

Mon, Apr. 23

In the examples on Friday, we saw that it did not matter which CW structure on S^2 we chose. In each case, we got the same answer, and these answers also agreed with the singular homology groups.

Theorem 5.36. *Let X be a space equipped with a choice of CW structure. Then*

$$\mathrm{H}_n^{cell}(X) \cong \mathrm{H}_n^{Sing}(X)$$

for all n.

Since the right-hand side does not depend on any choice of CW structure, the left-hand side must not either.

We do not give the proof (see Hatcher, Theorem 2.35). The idea is to first recognize that $\mathrm{H}_n^{Sing}(X) \cong \mathrm{H}_n^{Sing}(\mathrm{sk}_{n+1}X)$. Then we have

$$\mathrm{H}_n^{Sing}(X) \cong \mathrm{H}_n^{Sing}(\mathrm{sk}_{n+1}X) \twoheadleftarrow \mathrm{H}_n^{Sing}(\mathrm{sk}_n X) \longrightarrow \mathrm{H}_n^{Sing}(\mathrm{sk}_n X)/\mathrm{sk}_{n-1}X) \cong C_n^{cell}(X).$$

You show that this map lands in the subgroup $Z_n^{cell}(X)$ and induces an isomorphism to the quotient $Z_n^{cell}(X)/B_n^{cell}(X)$.

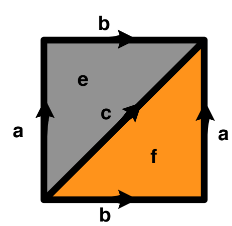
Example 5.37. Take $X = T^2$. The standard cell structure we have used has a single 0, two 1-cells a and b, and a single 2-cell e attached via $aba^{-1}b^{-1}$. Since there is a single 0-cell, this means that automatically $d_1 = 0$. To calculate $d_2(e)$, we wish to calculate the coefficient in front of a and b. For a, we must compose the attaching map $aba^{-1}b^{-1}$ with the projection onto the circle a. This means all of the b's are sent to 0, so in the end we have $aa^{-1} = 0$. The same goes for b, so $d_2 = 0$. The chain complex $C_*(T^2)$ is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}.$$

Since all differentials are zero in $C_*(T^2)$, it is immediate that

$$H_0(T^2) \cong \mathbb{Z}, \qquad H_1(T^2) \cong \mathbb{Z}^2, \qquad H_2(T^2) \cong \mathbb{Z}.$$

Example 5.38. (torus, second approach) Consider the CW structure on T^2 as given in the picture to the right. The resulting chain complex is



$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

We read off right away that $H_0(T^2) \cong \mathbb{Z}$. Then

$$H_1(T^2) = Z^3/\operatorname{im}(d_2) = \mathbb{Z}^3/\mathbb{Z}(1,1,-1) \cong \mathbb{Z}^2.$$

For the last isomorphism, note that since $(1,1,-1) \in \mathbb{Z}^3$ is linearly independent from $(0,1,0)$ and $(0,0,1)$, we can take these three elements as generators of the group $\mathbb{Z}^3$. It follows that the quotient is $\mathbb{Z}^2$. Finally,

$$H_2(T^2) = \ker(d_2) = \mathbb{Z}(1,1) \cong \mathbb{Z}.$$

There are a few algebraic results that are quite helpful in doing these computations.

Theorem 5.39. *(Fundamental theorem for finitely generated abelian groups) If A is a finitely generated abelian group, then*

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$$

for some non-negative integers r and k and positive integers $n_1, \ldots, n_k$.

Theorem 5.40. *(Smith normal form) Let A be an $n \times k$ matrix with integer values. Then, by using column and row operations, A can be reduced to*

$$A \sim \begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $n_i \mid n_{i+1}$. This is the **Smith normal form** *for the matrix.*

If a differential d_n is represented by A, then you reduce A to normal form, and the kernel of d_n will be (isomorphic to) $\mathbb{Z}^j$, where j is the number of zero columns in the normal form.

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Example 5.41. ($\mathbb{RP}^2$) We have a CW structure with a single cell in dimensions 0, 1, and 2. The attaching map for the 2-cell is $\gamma_2 : S^1 \longrightarrow S^1$. It follows that the chain complex $C_*(\mathbb{RP}^2)$ is

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Thus $H_0(\mathbb{RP}^2) \cong \mathbb{Z}$, $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, and $H_2(\mathbb{RP}^2) = 0$.

Example 5.42. (Klein bottle, first version) Recall that we have a CW structure on K having a single 0-cell and 2-cell and two 1-cells. The 2-cell is attached according to the relation $aba^{-1}b$. It follows that $C_*(K)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

We read off immediately that $H_0(K) \cong \mathbb{Z}$ and that $H_2(K) = 0$ since d_2 is injective. The remaining calculation is

$$H_1(K) = \mathbb{Z}^2/\mathbb{Z}(0,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Example 5.43. (Klein bottle, second version) Recall that we discussed a second CW structure on K having a single 0-cell and 2-cell and two 1-cells. The 2-cell is attached according to the relation c^2d^2. It follows that $C_*(K)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

We read off immediately that $H_0(K) \cong \mathbb{Z}$ and that $H_2(K) = 0$ since d_2 is injective. The remaining calculation is

$$H_1(K) = \mathbb{Z}^2/\mathbb{Z}(2,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Here the isomorphism $\mathbb{Z}^2/\mathbb{Z}(2,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is induced by the map

$$\mathbb{Z}^2 \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$
$$(n,k) \mapsto (n-k, k).$$

Example 5.44. (Orientable surfaces) We have a CW structure on M_g with a single 0-cell and 2-cell and $2g$ 1-cells. The attaching map for the 2-cell is the product of commutators $[a_1, b_1] \dots [a_g, b_g]$. It follows that $C_*(M_g)$ is the chain complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}.$$

So $H_0(M_g) \cong Z$, $H_1(M_g) \cong \mathbb{Z}^{2g}$, and $H_2(M_g) \cong \mathbb{Z}$.

Example 5.45. (Nonorientable surfaces) We have a CW structure on N_g with a single 0-cell and 2-cell and g 1-cells. The attaching map for the 2-cell is the product $a_1^2 \dots a_g^2$. It follows that $C_*(N_g)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}} \mathbb{Z}^g \xrightarrow{\ 0\ } \mathbb{Z}$$

So $H_0(N_g) \cong \mathbb{Z}$, $H_1(N_g) \cong \mathbb{Z}^g/\mathbb{Z}(2,\dots,2) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$, and $H_2(N_g) = 0$. Again, the isomorphism $\mathbb{Z}^g/\mathbb{Z}(2,\dots,2) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$ is induced by

$$\begin{aligned} \mathbb{Z}^g &\twoheadrightarrow \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} \\ (n_1, \dots, n_g) &\mapsto (n_1 - n_g, n_2 - n_g, \dots, n_{g-1} - n_g, n_g). \end{aligned}$$

Remark 5.46. According to the previous examples and our Proposition 3.52, a compact, connected surface M satisfies $H_2(M) \cong \mathbb{Z}$ if M is orientable and satisfies $H_2(M) = 0$ if M is not orientable.

So, for a surface, H_2 tells us about orientability.

We have seen that cellular homology tends to be quite computable, so what is the drawback? One major drawback is functoriality. Recall that any map of spaces $f : X \longrightarrow Y$ gave us a map on singular homology. For cellular homology, this is only true if the map is compatible with the CW structures, in the sense that f carries the n-skeleton of X into the n-skeleton of Y **for all** n. Such maps are called **cellular**, and this is a very strong condition. In fact, any map is homotopic to a cellular map, but in general finding a cellular approximation to a given map is quite nontrivial.

Example 5.47. Now let's consider $\mathbb{RP}^n$ for $n > 2$. The cellular chain complex is

$$\begin{array}{ccccccccc} C_n & \xrightarrow{1+(-1)^n} & C_{n-1} & \longrightarrow & \dots & \xrightarrow{\ 2\ } & C_1 & \xrightarrow{\ 0\ } & C_0 \\ \| & & \| & & & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z} & & & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

To understand the differential d_k, it suffices to understand what it does to the k-cell e_k. The attaching map for this k-cell is the double cover $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$. Then $d_k(e_k) = n_k e_{k-1}$, where n_k is the degree of the map

$$S^{k-1} \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}.$$

To visualize this, think of $\mathbb{RP}^{k-1}$ as the quotient of the northern hemisphere of S^{k-1} by a relation on the boundary. Then $\mathbb{RP}^{k-2}$ is the quotient of the boundary, so the quotient $\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ is the northern hemisphere with the equator collapsed. The map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ factors through $S^{k-1}/S^{k-2} \cong S^{k-1} \vee S^{k-1}$. The map on the nothern hemisphere $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$ is the identity. On the other hand, the map on the southern hemisphere can be identified with the map $(x_1, \dots, x_k) \mapsto (-x_1, \dots, -x_k)$. This is a homeomorphism, so the question is whether it is homotopic to the identity, in which case the map on this hemisphere corresponds to 1, or it is not, in which case the maps corresponds to -1. But this map is a sequence of k reflections, each of which has determinant -1. So the map has determinant $(-1)^k$. This number then agrees with the degree of the map, and we find that $n_k = 1 + (-1)^k$.

It follows that in degrees less than n we have

$$H_{2i}(\mathbb{RP}^n) = 0, i > 0, \qquad H_0(\mathbb{RP}^n) = \mathbb{Z}, \qquad H_{2i+1}(\mathbb{RP}^n) = \mathbb{Z}/2.$$

To determine $H_n(\mathbb{RP}^n)$, we consider $d_n : C_n \longrightarrow C_{n-1}$. If n is even, then d_n is injective, so $H_n(\mathbb{RP}^n) = 0$. On the other hand, if n is odd, then $d_n = 0$, so that $H_n(\mathbb{RP}^n) \cong \mathbb{Z}$.

The Euler characteristic computation according to homology is now

$$\chi(\mathbb{RP}^{2k}) = 0 + 0 + \cdots + 0 + 1 = 1, \qquad \chi(\mathbb{RP}^{2k+1}) = 1 + 0 + 0 + \cdots + 0 + 1 = 2.$$

Recall that we mentioned that for an n-manifold, the top homology group $H_n(M)$ is either $\mathbb{Z}$ or 0, depending on whether the manifold is orientable or not. The above shows that $\mathbb{RP}^n$ is orientable if and only if n is odd ($n \geq 1$).

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Example 5.48. We can also consider $X = \mathbb{CP}^n$. But this turns out to be much easier, since $\mathbb{CP}^n$ only has cells in even degrees. There can't possibly be any nonzero differentials! We then read off that

$$\mathrm{H}_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n \;\&\; k \text{ even} \\ 0 & \text{else.} \end{cases}$$

We also have $\chi(\mathbb{CP}^n) = n + 1$, and $\mathbb{CP}^n$ is always orientable.

Recall that we talked about the Euler characteristic for surfaces. For any chain complex C_*, we define the Euler characteristic of C_* by $\chi(C_*) = \sum (-1)^i \operatorname{rank}(C_i)$ (when this sum makes sense). Recall that the **rank** of a free abelian group is the maximal number of linearly independent elements. For example, if $C \cong \mathbb{Z}^r \oplus A$, where A is finite, then $\operatorname{rank} C = r$.

Lemma 5.49. *Suppose given a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of finitely-generated abelian groups. Then

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Proof. We show that $\operatorname{rank}(B) \geq \operatorname{rank}(A) + \operatorname{rank}(C)$ and leave the other direction as an exercise. Let $a_1, \ldots, a_r$ be a maximal linearly independent set in A and $c_1, \ldots, c_s$ a maximal linearly independent set in C. Since g is surjective, we can lift these elements to $\tilde{c}_i \in B$. We claim that the collection $\{f(a_i)\} \cup \{\tilde{c}_j\}$ is linearly independent. Thus consider an equation

$$\sum_i m_i f(a_i) + \sum_k n_k \tilde{c}_k = 0.$$

By applying g, we get

$$\sum_k n_k c_k = 0.$$

Since the c_k's are independent, we conclude that $n_k = 0$ for all k. Since f is injective, we now learn that

$$\sum_i m_i a_i = 0.$$

But since the a_i's are independent, we learn that $m_i = 0$ for all i. This shows that $\{f(a_i)\} \cup \{\tilde{c}_j\}$ is independent. ■

Proposition 5.50. *For any chain complex, we have $\chi(C_*) = \chi(H_*(C_*))$.*

Proof. The key is to note that we have short exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0.$$

and

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0.$$

By a Lemma 5.49, these tell us that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$

and

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i).$$

So

$$\sum_i (-1)^i \operatorname{rank}(C_i) = \sum_i (-1)^i (\operatorname{rank}(B_i) + \operatorname{rank}(H_i) + \operatorname{rank}(B_{i-1})).$$

This is a telescoping sum, and we end up with $\chi(H_*)$. ■

As an example, we talked about the homology of $\mathbb{RP}^2$ earlier. We saw this was

$$H_0(\mathbb{RP}^2) \cong \mathbb{Z}, \qquad H_1(\mathbb{RP}^2) = \mathbb{Z}/2, \qquad H_2(\mathbb{RP}^2) = 0.$$

Since the standard model for $\mathbb{RP}^2$ has no cells above dimension 2, there is of course no homology in higher dimensions. The Euler characteristic computation according to homology is

$$\chi(\mathbb{RP}^2) = \operatorname{rank}(\mathbb{Z}) - \operatorname{rank}(\mathbb{Z}/2) = 1.$$

Proposition 5.50 tells us that the Euler characteristic only depends on the homology of the space, not on the particular cellular model.