

Math 751 – Spring 2024

Chromatic Homotopy Theory

Worksheet 4

1. Show that a map $f: X \rightarrow Y$ is a rational equivalence if and only if the induced map $f_Q: X_Q \rightarrow Y_Q$ is a stable equivalence.
2. Show that if X is a spectrum and Z is a \mathbb{Q} -local spectrum, then any morphism $f: X \rightarrow Z$ in \mathbf{HoSp} factors uniquely through the rationalization X_Q .
3. In this problem, you will show that \mathbf{HoSp}_Q is equivalent to $HQ\text{-Mod}$, i.e. the category of modules over the h -ring HQ .
 - (a) Recall that \mathbf{HoSp}_Q means the full subcategory of \mathbf{HoSp} on \mathbb{Q} -local spectra. Show that every \mathbb{Q} -local spectrum inherits a canonical structure of HQ -module.
 - (b) Show that every map of \mathbb{Q} -local spectra is automatically a map of HQ -modules.
4. Let A be an abelian group, and let $e: A \rightarrow A$ be an idempotent.
 - (a) Show that $(\text{id} - e)$ is also an idempotent and that $\text{im}(\text{id} - e)$ is the same as $\ker e$.
 - (b) Show that A is isomorphic to $\text{im}(e) \oplus \text{im}(\text{id} - e)$.
 - (c) Show that $\text{colim} \left(A \xrightarrow{e} A \xrightarrow{e} A \xrightarrow{e} \dots \right)$ is isomorphic to $\text{im}(e)$.