STABLE $\mathbb{A}^1$-HOMOTOPY THEORY

BERTRAND GUILLOU

1. Introduction

Recall from the previous talk that we have our category pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}o_{\mathbb{A}^1,\bullet}(k)$ over a field k. We will often refer to an object of our category (i.e. a simplicial sheaf on Sm_k) simply as a space.

2. The Stable $\mathbb{A}^1$-homotopy category of T-spectra

Definition 1. A T-prespectrum E is a sequence E_n of pointed spaces together with structure morphisms $\sigma_n : E_n \wedge T \to E_{n+1}$ for all n. A morphism $\lambda : E \to F$ of T-prespectra is a sequence of maps $\lambda_n : E_n \to F_n$ commuting with the structure morphisms. We will denote the category of T-prespectra by $PSp_T(k)$.

Example 1. For any based space $\mathcal{X}$, we have the *suspension prespectrum* $\Sigma^\infty(\mathcal{X})$ of $\mathcal{X}$ given by $\Sigma^\infty(\mathcal{X})_n = \mathcal{X} \wedge T^{(n)}$. The structure morphisms are simply the association isomorphisms.

Let $p, q \in \mathbb{Z}$. For any $E \in PSp_T(k)$ and $U \in \mathrm{Sm}_k$, we define the stable homotopy group of degree p and weight q to be the presheaf of groups given on some $U \in \mathrm{Sm}_k$ by

$$\pi_{p,q}(E)(U) = \operatorname*{colim}_r [S_s^{p-q+r} \wedge \mathbb{G}_m^{q+r} \wedge (U_+), E_n].$$

The above definition relies, of course, on the fact that $S_s^1 \wedge \mathbb{G}_m \simeq T$.

Definition 2. A morphism $f : E \to F$ in $PSp_T(k)$ is said to be a stable $\mathbb{A}^1$-weak equivalence if it induces an isomorphism of presheaves of stable homotopy groups

$$\pi_{p,q}(E) \xrightarrow{\cong} \pi_{p,q}(F)$$

for all $p, q \in \mathbb{Z}$. The $\mathbb{A}^1$ stable homotopy category $\mathcal{S}H(k)$ is then obtained from $PSp_T(k)$ by inverting the stable $\mathbb{A}^1$-weak equivalences.

As in the classical case, $\mathcal{S}H(k)$ is a triangulated category (the triangulation is given by simplicial suspension), and T-suspension induces an equivalence on $\mathcal{S}H(k)$. The adjoint of T-suspension is given by Ω_T. This functor Ω_T is defined levelwise; that is, $(\Omega_T E)_n = \Omega_T(E_n)$. It remains to describe the T-loops functor on spaces. But this is of course simply the internal hom $\underline{\mathrm{Hom}}(T, E_n)$ of pointed presheaves given on some $U \in \mathrm{Sm}_k$ by

$$\Omega_T(E_n)(U) = \underline{\mathrm{Hom}}(T, E_n)(U) = \mathrm{Hom}(T \wedge U_+, E_n).$$

One of the technical details needed in the proof that these induce an adjoint equivalence is the following lemma of Voevodsky:

Lemma 1. *The cyclic permutation of factors in T^3 induces the identity map in $\mathcal{H}o_{\mathbb{A}^1,\bullet}(k)$.*

Proof. Recall from last time that, on the level of Nisnevich sheaves, we have an isomorphism $T^3 \cong A^3/(A^3 - 0)$. Now we have an action of Gl_3 on $\mathbb{A}^3$, and we have a map

$$(Gl_3)_+ \wedge A^3/(A^3 - 0) \to A^3/(A^3 - 0).$$

The cyclic permuation of factors on T^3 corresponds to the cyclic permutation of coordinates on A^3, and this automorphism comes from $Sl_3 \subset Gl_3$. But now recall that, over a field, $Sl_n = E_n$; that is, Sl_n is generated by elementary matrices. Given any elementary matrix $E_{i,j}(\alpha)$, there is an obvious $\mathbb{A}^1$-path in Sl_n to the identity, namely $t \mapsto E_{i,j}(t\alpha)$. This implies that any automorphism of $\mathbb{A}^n$ coming from Sl_n is the identity in the homotopy category. ■

In fact, if we take any based space Y which is (some suspension of) a smooth scheme, we can form the category of Y-spectra, and the resulting stable homotopy category will be triangulated if Y satisfies the above lemma. In particular, this works for $Y = S^1_s$ or $\mathbb{G}_m$.

3. Good categories of spectra

The constructions of good categories of spectra have analogues in the motivic world. In particular, Rick Jardine has developed the theory of motivic symmetric spectra, and Po Hu has given a theory of S-modules. We will discuss these briefly.

3.1. Symmetric Spectra

Definition 3. A symmetric T-spectrum is a T-prespectrum E as defined above together with symmetric group actions $\Sigma_n \hookrightarrow \mathrm{Hom}(E_n, E_n)$ such that the composite structure maps $E_n \wedge T^{\wedge k} \to E_{n+k}$ are $\Sigma_n \times \Sigma_k$-equivariant.

An obvious example of a symmetric spectrum is the sphere spectrum $\Sigma^\infty_T S^0$. We will see that the Eilenberg-MacLane spectrum is also naturally a symmetric spectrum.

Just as in the topological case, we can then define a smash product of two symmetric spectra E and F. First, if we forget about the structure maps, then a symmetric T-spectrum is just a symmetric sequence of based spaces. There is an obvious "external" tensor product of symmetric sequences: $(E \otimes F)_{(m,n)} = E_m \wedge F_n$. One then obtains an internal tensor product by left Kan extension along $(m, n) \mapsto m+n$. Note that the structure of a symmetric T-spectrum E is just a map $m_E : E \otimes \Sigma^\infty S^0 \to E$ of symmetric sequences.

Finally, the smash product $E \wedge F$ is defined as the coequalizer of symmetric sequences

$$E \otimes F \otimes \Sigma^\infty S^0 \rightrightarrows E \otimes F \longrightarrow E \wedge F,$$

where we coequalize the maps $1_E \otimes m_F$ and $m_E \otimes 1_F \circ 1_E \otimes \tau$.

This construction now gives a symmetric monoidal structure on the category of motivic symmetric spectra. Moreover, Jardine has put a model structure on this category which gives a homotopy category equivalent to $\mathcal{SH}(k)$.

3.2. S-modules

To define S-modules, one needs to work with coordinate-free spectra; that is, one works with spectra indexed over finite-dimensional subspaces of some infinite-dimensional real inner product space. In the motivic context, there is generally no such universe on which to index spectra. Hu has to do quite a bit of work to circumvent this difficulty. In particular, she indexes her spectra on subspaces of finite codimension inside some infinite dimensional k-vector space.

There is not much trouble in defining the spaces of the linear isometries operad; one essentially defines these using limits and colimits of Stiefel varieties. The twisted half-smash product also takes quite a bit of work to construct, but once these tools are in place, almost everything simply carries over from EKMM. Hu then puts a model structure on this category of S-modules and shows it to give the same homotopy category.

4. Examples

Given any spectrum E, we obtain a cohomology theory on our category of spaces by setting

$$E^{p,q}(X) = [\Sigma_T^\infty X_+, S_s^{p-q} \wedge \mathbb{G}_m^q \wedge E].$$

We list below a few examples of spectra which give rise to cohomology theories of interest.

4.1. Eilenberg-MacLane spectrum

What should the Eilenberg-MacLane spectrum be in this context? In topology, the Eilenberg-MacLane spectrum $H_\mathbb{Z}$ is given by $(H_\mathbb{Z})_n = K(\mathbb{Z}, n)$. The structure maps $\Sigma K(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1)$ are given by the adjoints of equivalences $K(\mathbb{Z}, n) \to \Omega K(\mathbb{Z}, n+1)$. To construct Eilenberg-MacLane spaces, we can for example use the Dold-Kan equivalence between simplicial abelian groups and nonnegative chain complexes.

Unfortunately, it is not so obvious how to construct Eilenberg-MacLane spaces here. There is another way of constructing $K(\mathbb{Z}, n)$'s in topology, by use of the Dold-Thom theorem.

Theorem 1 (Dold-Thom). *For a pointed CW complex X, there is a weak equivalence*

$$Symm^\infty(X, *)^+ \simeq \prod_{n \geq 0} K(H_n(X), n),$$

where $Symm^\infty(X, *)$ denotes the infinite symmetric product of X and the $+$ denotes the group completion.

Proof. There are a number of ways to prove this. One way is to simply show that the functor $(X, *) \mapsto \pi_*(Symm^\infty(X, *)^+)$ satisfies the axioms of a reduced homology theory and to note that it certainly satisfies the dimension axiom.

Alternately, $Symm^\infty(X, *)^+$ is some kind of free abelian group functor on spaces. There is a canonical map

$$\mathbb{N}[S_\bullet(X)] \to S_\bullet(Symm^\infty(X, *)),$$

where $\mathbb{N}[-] : sSet \to sAbMon$ is the free abelian monoid functor on simplicial sets and $S_\bullet = \mathrm{Hom}(\Delta^\bullet, -)$ is the singular set functor. It is not difficult to see that for any simplicial set K, the map $|\mathbb{N}[K]| \to Symm^\infty(|K|)$ is a weak homotopy equivalence, and it follows that the above map is a weak equivalence. This map then extends via naive group completion to a weak equivalence

$$\mathbb{Z}[S_\bullet(X)] \to S_\bullet(Symm^\infty(X, *))^+$$

of simplicial abelian groups. On the other hand, $|S_\bullet(Symm^\infty(X, *))^+|$ is a (topological) group completion of $|S_\bullet(Symm^\infty(X, *))| \simeq Symm^\infty(X, *)$. It follows that we can pull the $+$ inside to get a weak equivalence

$$\mathbb{Z}[S_\bullet(X)] \to S_\bullet(Symm^\infty(X, *)^+).$$

But now, by the Dold-Kan equivalence, we have an isomorphism

$$\pi_i(\mathbb{Z}[S_\bullet(X)]) \cong H_i(N(\mathbb{Z}[S_\bullet(X)])) = H_i(X; \mathbb{Z}),$$

where $N : sAbGp \to Ch_{\geq 0}(\mathbb{Z})$ is the normalized chain complex functor. ■

Note that applying this theorem to the case where X is a Moore space of type $M(G, n)$ implies that $Symm^{\infty}(X)$ is a $K(G, n)$.

Now we need some analogue of the infinite symmetric product in our category of spaces, and it turns out that the right thing to use is the "free presheaf with transfers" functor

$$\mathbb{Z}_{tr} : Pre_*(\mathrm{Sm}_k) \to AbPre(\mathrm{Sm}_k).$$

We define this on a smooth scheme X by setting $\mathbb{Z}_{tr}(X)(Y)$ to be the free abelian group on cycles $Z \subset Y \times X$ which are finite and surjective over Y. We then define $\mathbb{Z}_{tr}$ on $Pre_*(\mathrm{Sm}_k)$ to be the unique colimit-preserving extension.

Let $\mathbb{Z}_{tr}^{eff}(X)$ be the subpresheaf defined by taking the free abelian monoid rather than the free abelian group. The following theorem justifies $\mathbb{Z}_{tr}$ as a replacement for the infinite symmetric product:

Theorem 2 (Suslin-Voevodsky)**.** *Let* $\mathrm{char}(k) = p$. *There is an isomorphism*

$$\mathbb{Z}_{tr}^{eff}(X)(S)[1/p] \xrightarrow{\cong} \mathrm{Hom}\left(S, \coprod_{n=0}^{\infty} S^n(X)\right)[1/p]$$

for any normal, connected S.

Note that there is a canonical map $X \to \mathbb{Z}_{tr}(X)$, and we have product maps $\mathbb{Z}_{tr}(X) \wedge \mathbb{Z}_{tr}(Y) \xrightarrow{\mu} \mathbb{Z}_{tr}(X \wedge Y)$ given by external products of cycles.

Now we are ready to define our Eilenberg-MacLane spectrum. We set

$$K(\mathbb{Z}, n) = \mathbb{Z}_{tr}(T^{\wedge n}) \cong \mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - 0).$$

The structure maps are defined as

$$K(\mathbb{Z}, n) \wedge T \to \mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(A^n - 0) \wedge \mathbb{Z}_{tr}(\mathbb{A}^1)/\mathbb{Z}_{tr}(\mathbb{A}^1 - 0) \xrightarrow{\mu} \mathbb{Z}_{tr}(\mathbb{A}^{n+1})/\mathbb{Z}_{tr}(\mathbb{A}^{n+1} - 0) = K(\mathbb{Z}, n+1).$$

That this Eilenberg-MacLane spectrum is an Ω-spectrum (the adjoints to the structure maps are equivalences) is given by Voevodsky's Cancellation Theorem.

Remark 1. The cohomology theory represented by this spectrum is known as motivic cohomology, and we are to think of it as an analogue of ordinary cohomology. On the other hand, one should not take the analogy too seriously: motivic cohomology does not satisfy the dimension axiom. In fact, the motivic cohomology of $\mathrm{Spec}(k)$ is mostly unknown. We have that $H^{p,q}(\mathrm{Spec}(k), \mathbb{Z}) = 0$ if either

- $p > q$ or
- $q < 0$ or
- $q = 0$ and $p \neq 0$ or
- $q = 1$ and $p \neq 1$.

Moreover, $H^{0,0}(\mathrm{Spec}(k), \mathbb{Z}) = \mathbb{Z}$, $H^{1,1}(\mathrm{Spec}(k), \mathbb{Z}) = F^{\times}$, and more generally $H^{n,n}(\mathrm{Spec}(k), \mathbb{Z}) = K_n^M(k)$.

4.2. Algebraic K-theory spectrum

Let $BGl_n = Gr(n, \infty)$ be the Grassmannian (defined as the colimit of the schemes $Gr(n, m)$). Then let $BGl = \mathrm{colim}_n BGl_n$. The algebraic K-theory spectrum K is given by $K_n = BGl$ and structure morphisms $BGl \wedge T \simeq BGl \wedge \mathbb{P}^1 \to BGl$ are given by the classifying map for the "Bott element" $1 - H$. (H is the dual of the canonical line bundle on $\mathbb{P}^1$.)

4.3. Algebraic Cobordism spectrum

The Algebraic Cobordism spectrum is defined in almost complete analogy to MU. As above, we have the classifying space BGl_n, and over this space we have the universal n-plane bundle E_n. We then define the algebraic cobordism spectrum MGl by $MGl_n = T(E_n)$. The structure maps are defined as follows:

$$MGl_n \wedge T = T(E_n) \wedge T \cong T(E_n \oplus \mathcal{O}^1) \xrightarrow{T(\lambda_n)} T(E_{n+1}) = MGl_{n+1},$$

where $\lambda_n : E_n \oplus \mathcal{O}^1 \to E_{n+1}$ is the classifying map of the bundle.

5. $\pi_0^s(S^0)$

First we need to introduce the Grothendieck-Witt ring of a field k, where the characteristic of k is not 2. Consider the free abelian group on all isomorphism classes of pairs (V, μ), where V is a finite-dimensional k-vector space and μ is a nondegenerate symmetric bilinear form on V. Then identify $(V \oplus V', \mu \oplus \mu')$ with the sum in the group. This becomes a commutative ring, with multiplication given by tensor product.

However, if the characteristic of k is not equal to 2, then every symmetric bilinear form is diagonalizable, and so every class (V, μ) splits into a sum of one-dimensional pieces (V_i, μ_i). Of course, each μ_i is just given by an element of $k^\times$. Thus $GW(k)$ is generated as an algebra by symbols $\langle u \rangle$, with $u \in k^\times$. Change of basis by multiplying by $a \in k^\times$ shows that we have a relation $\langle u \rangle = \langle ua^2 \rangle$.

Now given $u \in k^\times$, we want to construct an element of $[S^0, S^0]$ (this is the endomorphism ring of the sphere spectrum $\Sigma_T^\infty \operatorname{Spec}(k)_+$). An element of $k^\times$ can be regarded as a map $u : \operatorname{Spec}(k) \to \mathbb{G}_m$. We also have the "Hopf map" $\eta : \mathbb{G}_m \to S^0$ given as follows. We have the classical Hopf map $\mathbb{A}^2 - 0 \to \mathbb{P}^1$. But $\mathbb{P}^1 \simeq T$ and $\mathbb{A}^2 - 0 \simeq T \wedge \mathbb{G}_m$ by purity. Now we define an element in $[\mathbb{G}_m, \mathbb{G}_m]$ by $\langle u \rangle = 1 + \eta \cdot u$.

It turns out that the simplicial suspension of this map, as an element of $[\mathbb{P}^1, \mathbb{P}^1]$ has the simpler description as the map $[x_0 : x_1] \mapsto [x_0 : ux_1]$. Either way, we obtain an element of $[S^0, S^0]$. Of course, one must check that this is well defined as a map from $GW(k)$. The Hopkins-Morel result is then

Theorem 3. *If k is a perfect field, and* $\operatorname{char}(k) \neq 2$, *the above map*

$$GW(k) \to [S^0, S^0]$$

is an isomorphism.