WOMP 2004: CATEGORY THEORY

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Category theory has been around for quite some time now and pervades modern mathematics; it is not really an area of mathematics so much as an area of meta-mathematics. It describes frameworks in which mathematics can be, and usually is, done.

More than that, in many cases category theory has encouraged, or at least facilitated, a change of perspective. When one studies mathematics, one often wants to study mathematical objects: sets, groups, topological spaces, Lie groups, etc. In category theory, however, at least as much emphasis is placed on maps between objects as on the objects themselves, and in many cases it turns out to be much more fruitful to make the maps the center of focus.

Another great success of category theory has been to make ideas like "canonical" precise. It was finally through the use of "natural transformations" that mathematicians were able to make this notion rigorous.

We will try to go through some of the basic definitions and provide lots of examples to give a feel for what categories can look like.

1 Categories

Definition. A *category* $\mathscr{C}$ is a collection of "objects", denoted $Ob(\mathscr{C})$, together with, for each pair of objects $X, Y \in Ob(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of "morphisms" which satisfies the following:

- For each $X, Y, Z \in Ob(\mathscr{C})$, there is a "composition" function

$$\circ : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z).$$

  We write $g \circ f$ or gf for $\circ(g, f)$.
- For each $X \in Ob(\mathscr{C})$ there exists a distinguished element $1_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that for any $Y, Z \in Ob(\mathscr{C})$ and $f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $g \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ we have

$$1_X \circ f = f \qquad \text{and} \qquad g \circ 1_X = g.$$

- Composition is associative, i.e., $h(gf) = (hg)f$.

Remark. We often write $\mathscr{C}(X, Y)$ for $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and we often write $X \in \mathscr{C}$ for $X \in Ob(\mathscr{C})$. Morphisms are often called arrows.

Remark. A category $\mathscr{C}$ is called *small* if the collection $Ob(\mathscr{C})$ of objects forms a set.

Categories abound in mathematics. Here are just a few of the more common examples.

Example 1.

(a) **1**: the trivial category. It has exactly one object, $*$, and exactly one morphism, 1_*.

(b) **0**: the really trivial category. It has an empty set of objects and an empty set of morphisms.

(c) **Set**: the objects are sets and the morphisms are functions.

(d) **FinSet**: the objects are finite sets and morphisms are functions.

(e) $\mathbf{Vect_k}$, where k is a field: the objects are vector spaces over k and morphisms are k-linear homomorphisms.

(f) $(\mathbf{Vect}_k)_{\text{f.d.}}$, where k is a field: the objects are finite-dimensional vector spaces and morphisms are k-linear homomorphisms.

(g) **Gp**: the objects are groups and the morphisms are homomorphisms.

(h) **AbGp**: the objects are abelian groups and the morphisms are homomorphisms.

(i) $\mathbf{Mod}_R$, where R is a commutative ring: the objects are R-modules and the morphisms are R-module homomorphisms.

(j) **Top**: the objects are topological spaces and the morphisms are continuous maps.

(k) $\mathbf{Top}_*$: the objects are pointed topological spaces (spaces with a distinguished base point) and the morphisms are basepoint-preserving continuous maps.

(l) Let G be any group. We can then regard G as a category as follows: the category $\mathscr{G}$ has only one object $\star$, and $\text{Hom}_{\mathscr{G}}(\star, \star) = G$. Composition of morphisms is defined by the group operation of the group G.

Most of the examples above are of the same flavor: they provide frameworks in which a mathematician might want to work. On the other hand, there are other sorts of categories which are useful to consider. For instance, consider the pictorial diagram

$$X \xrightarrow{\alpha} Y.$$

We can think of this as representing a category. The objects are "X" and "Y". There is a morphism α from X to Y. This almost describes the category, but remember that in any category there must be an identity arrow for each object. Thus we must throw in arrows 1_X and 1_Y. The last thing to do is to describe how all of the arrows compose, but in fact there is no choice. Any time we want to compose two arrows in this category, at least one of them is necessarily an identity arrow, and so we know what the composition must be. Similarly, you can imagine what categories the diagrams

$$A \leftarrow B \rightarrow C \qquad \text{and} \qquad s \rightrightarrows t$$

represent.

Definition. Given a category $\mathscr{C}$, a subcategory $\mathscr{B} \subset \mathscr{C}$ is just what one would guess: the objects of $\mathscr{B}$ form a subcollection of the objects of $\mathscr{C}$; for $X, Y \in \mathscr{B}$ the morphisms from X to Y are a subset of the morphisms in $\mathscr{C}$; and the composition in $\mathscr{B}$ is the composition in $\mathscr{C}$. We say that $\mathscr{B}$ is a *full subcategory* if for all $X, Y \in \mathscr{B}$ we have $\mathrm{Hom}_{\mathscr{B}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Example 2. Of the categories we have already seen, **AbGp**, **FinSet**, and $(\mathbf{Vect}_k)_{\text{f.d.}}$ are full subcategories of **Gp**, **Set**, and $\mathbf{Vect}_k$, respectively.

Example 3. Given any category $\mathscr{C}$, we can form a subcategory $\mathscr{B} \subset \mathscr{C}$ that has the same objects and such that the morphism from X to Y in $\mathscr{B}$ are exactly the isomorphisms from X to Y in $\mathscr{C}$. Note that this still defines a category since 1_X is an isomorphism for all X and since isomorphisms are closed under composition. Of course, such a subcategory $\mathscr{B}$ is usually not full.

For example, we have the category of topological spaces and homeomorphisms sitting inside the category of topological spaces and continuous maps. This is certainly not full.

1.1 Iso's, Mono's, and Epi's

In the categories we have described above, we know what it means for two objects to be "the same". In **Set**, this means that they are bijective; in **Top**, it means they are homeomorphic. In fact, all of these fall under the general categorical notion of isomorphism:

Definition. Let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. We say that f is an *isomorphism* if there exists a morphism $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.

Exercise 1. Show that a function $X \to Y$ of sets is a bijection if and only if it is an isomorphism as defined above.

Thus in **Set**, a morphism (function) is an isomorphism (bijection) if and only if it is both injective and surjective. The notions of injections and surjections generalize to arbitrary categories, but it is no longer true that the isomorphisms are the morphisms which are both injective and surjective.

Definition. Let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. We say that f is a *monomorphism* if for every $Z \in \mathscr{C}$, the induced map of sets

$$f_* : \mathrm{Hom}_{\mathscr{C}}(Z, X) \to \mathrm{Hom}_{\mathscr{C}}(Z, Y)$$

is an injection(of sets). We say f is an *epimorphism* if

$$f^* : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$$

is an injective map of sets for each $Z \in \mathscr{C}$.

Exercise 2. Show that if $g \circ f$ is a monomorphism then so is f. Similarly, if $g \circ f$ is an epimorphism then so is g. Conclude that any isomorphism is necessarily both a monomorphism and an epimorphism.

Exercise 3.

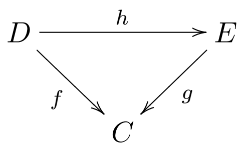
(a) In **Top**, a map is a monomorphism if and only if it is injective and an epimorphism if and only if it is surjective.

(b) In the category **Haus** of Hausdorff spaces and continuous maps, a map is a monomorphism if and only if it is injective and an epimorphism if and only if it has dense image. Thus the inclusion $U \hookrightarrow X$ of a dense subset is both a monomorphism and an epimorphism but not necessarily an isomorphism.

(c) The ring homomorphism $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism in **Comm**, though it is not an isomorphism.

1.2 Under and Over-Categories

Definition. Let $\mathscr{C}$ be a category and $C \in \mathscr{C}$ any object. We form the category $(\mathscr{C} \downarrow C)$ of *objects over* C as follows: an object is an object D of $\mathscr{C}$ together with a given morphism $D \to C$. If (D, f) and (E, g) are objects over C, a morphism $h : (D, f) \to (E, g)$ is defined to be a morphism $h : D \to E$ which makes the following diagram commute:

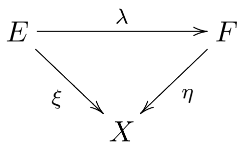


Similarly, we define the category $(C \downarrow \mathscr{C})$ of *objects under* C to have, as objects, objects D of $\mathscr{C}$ with a morphism $C \to D$; morphisms are defined analogously.

Example 4.

(a) Let $* \in$ **Top** be a one-point set. Then $(* \downarrow$ **Top**$)$ is just $\mathbf{Top}_*$, the category of pointed spaces.

(b) Let X be a topological space. We define $\mathbf{Vect}_X$ to be the category of vector bundles over X. That is, an object is a vector bundle $E \xrightarrow{\xi} X$, and a morphism is just a morphism of bundles:



Over and under-categories are examples where morphisms really become the central object of study; indeed, the objects of this category are morphisms in the category $\mathscr{C}$.

2 Functors

In this section, we will discuss functors, which are "maps between categories."

Definition. Let $\mathscr{C}$ be any category. We define the *opposite category* $\mathscr{C}^{op}$ to be the category with the same objects as $\mathscr{C}$ and with

$$\mathrm{Hom}_{\mathscr{C}^{op}}(X,Y) := \mathrm{Hom}_{\mathscr{C}}(Y,X).$$

That is, we merely switch all of the directions of the arrows in $\mathscr{C}$. Composition in $\mathscr{C}^{op}$ is induced from composition in $\mathscr{C}$. Note that if morphisms in $\mathscr{C}$ correspond to functions (with possibly extra structure) then morphisms in $\mathscr{C}^{op}$ will not correspond to functions.

Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *(covariant) functor* $F : \mathscr{C} \to \mathscr{D}$ is the following data: for each $C \in \mathscr{C}$ we have an object $F(C) \in \mathscr{D}$, and for each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$ we have an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(C), F(C'))$ such that

$$F(1_C) = 1_{F(C)} \qquad \text{and} \qquad F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* $F : \mathscr{C} \to \mathscr{D}$ is merely a (covariant) functor $F : \mathscr{C}^{op} \to \mathscr{D}$. Thus a covariant functor is one which preserves the directions of the arrows, and a contravariant functor is one which reverses the directions of the arrows.

Remark. If $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor and f is an arrow in $\mathscr{C}$, we often write f_* for $F(f)$. If F is contravariant, we write f^* for $F(f)$.

Example 5.

(a) Let $M \in \mathbf{Mod}_R$ for some commutative ring R. Then M defines a functor $\mathrm{Hom}_R(M, -) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ defined on objects by $\mathrm{Hom}_R(M,-)(N) = \mathrm{Hom}_R(M,N)$. If $f : N \to P$ is an R-module homomorphism, $f_* : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,P)$ is defined by $f_*(\varphi) = f \circ \varphi$.

(b) As above, let $M \in \mathbf{Mod}_R$ for some commutative ring R. We can then define a functor $(-) \otimes_R M : \mathbf{Mod}_R \to \mathbf{Mod}_R$ defined on an object N to be $N \otimes_R M$. Given a morphism $\varphi : N \to P$, we define $\varphi \otimes_R M : N \otimes_R M \to P \otimes_R M$ to be $\varphi \otimes 1_M$. That is,

$$\varphi \otimes_R M \left(\sum_i n_i \otimes m_i \right) := \sum_i \varphi(n_i) \otimes m_i.$$

(c) Let **Comm** denote the category of commutative rings with unit. For any topological space X, we denote by $\Gamma(X)$ the set of continuous $\mathbb{C}$-valued functions on X. Note that $\Gamma(X)$ is a commutative ring with unit, where the sum and product are defined pointwise. Then Γ defines a contravariant functor $\mathbf{Top} \to \mathbf{Comm}$. If $f : X \to Y$ is a continuous map of topological spaces, $\Gamma(f) : \Gamma(Y) \to \Gamma(X)$ is defined by $\Gamma(f)(\lambda) = \lambda \circ f$.

(d) For any $n \geq 1$, Gl_n defines a functor $\mathbf{Comm} \to \mathbf{Gp}$. On objects, $Gl_n(R)$ is just the group of $n \times n$ invertible matrices over the ring R. Given a ring homomorphism $\varphi : R \to S$, $Gl_n(\varphi)$ is defined entrywise; that is, $Gl_n(\varphi)$ takes a matrix over R with entries a_{ij} to the matrix over S with entries $\varphi(a_{ij})$.

(e) There is a “forgetful” functor $U : \mathbf{Gp} \to \mathbf{Set}$ which takes a group to its set of elements and which takes a homomorphism to the underlying function. Essentially, this functor takes groups and “forgets” the extra group structure and remembers only the underlying set. There is similarly a forgetful functor $\mathbf{AbGp} \to \mathbf{Set}$.

(f) In the example above, we saw functors which forget extra structure, but often there are functors going in the opposite direction which build in the extra structure. Such functors are called “free” functors. For example, there is the free abelian group functor $F : \mathbf{Set} \to \mathbf{AbGp}$ defined on objects by

$$F(X) = \bigoplus_{x \in X} \mathbb{Z}.$$

An element of $F(X)$ is a finite formal $\mathbb{Z}$-linear combination of elements of X, and the group operation is defined by

$$\left(\sum_{x\in X} n_x x\right) \cdot \left(\sum_{x\in X} m_x x\right) := \sum_{x\in X} (n_x + m_x)x.$$

Given a function $f : X \to Y$, $F(f)$ is defined by

$$F(f)\left(\sum_{x\in X} n_x x\right) := \sum_{x\in X} n_x f(x).$$

(g) Let G be a group; as we saw above, we can regard G as a category $\mathscr{G}$ with one object. Then a functor $F : \mathscr{G} \to \mathbf{Set}$ is exactly the same data as a G-set, i.e., a set with an action of G.

(h) For those who have seen fundamental groups before, we have the functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Gp}$ which assigns to a space X with basepoint x the fundamental group of loops based at x, $\pi_1(X, x)$. Given a basepoint-preserving map of pointed spaces $f : X \to Y$, the homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is defined by sending the class of a loop α to the class of the loop $f \circ \alpha$.

Exercise 4. Show that if $M \in \mathbf{Mod}_R$ for some commutative ring R then $\mathrm{Hom}_R(-, M)$ determines a contravariant functor $\mathbf{Mod}_R \to \mathbf{Mod}_R$.

Exercise 5. If $\mathscr{G}$ and $\mathscr{H}$ are groups regarded as categories, characterize functors $\Phi : \mathscr{G} \to \mathscr{H}$.

Exercise 6. There are also forgetful functors

$$\mathbf{Top} \to \mathbf{Set}, \quad \mathbf{Top}_* \to \mathbf{Top}, \quad \mathbf{Mod}_R \to \mathbf{AbGp}, \quad \mathbf{Comm} \to \mathbf{AbGp}.$$

In each of these cases there are “free” functors going in the opposite direction. What are they?

Definition. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be *faithful* if for all $C, C' \in \mathscr{C}$ the function $F(-)$: $\mathrm{Hom}_{\mathscr{C}}(C, C') \to \mathrm{Hom}_{\mathscr{D}}(F(C), F(C'))$ is injective; F is said to be *full* if each $F(-)$ is surjective. We say F is *fully faithful* if it is both full and faithul.

Example 6. If $\mathscr{B} \subset \mathscr{C}$ is a subcategory, then the obvious inclusion functor $i : \mathscr{B} \to \mathscr{C}$ is faithful. This functor is fully faithful if and only if $\mathscr{B}$ is a full subcategory.

Exercise 7. Show that the forgetful functors we have seen above are all faithful.

Exercise 8. Find examples of M and R such that $(-) \otimes_R M$ is not full or not faithful.

3 Natural Transformations

We have described categories and functors, which are maps between categories. Now we will describe natural transformations, which are maps between functors.

Definition. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A natural transformation $\eta : F \to G$ is a collection of maps $\eta_C : F(C) \to G(C)$, one for each $C \in \mathscr{C}$, such that for any $C, C' \in \mathscr{C}$ and any $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$, the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_{C'}} \\ G(C) & \xrightarrow{G(f)} & G(C') \end{array}$$

Example 7.

(a) Let $M, N \in \mathbf{Mod}_R$ for some commutative ring R and let $f : M \to N$ be an R-module homomorphism. Then f defines a natural transformation $\eta : (-) \otimes_R M \to (-) \otimes_R N$ defined by $\eta_P = 1_P \otimes f : P \otimes_R M \to P \otimes_R N$. Similarly, f also defines a natural transformation $\mathrm{Hom}_R(-, M) \to \mathrm{Hom}_R(-, N)$.

(b) Let G be a group and X, Y be G-sets. As we saw above, we can regard X and Y as coming from functors $\mathcal{X}, \mathcal{Y} : \mathscr{G} \to \mathbf{Set}$. Suppose that $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation. Since $\mathscr{G}$ has only one object $\star$, and $\mathcal{X}(\star) = X$, $\mathcal{Y}(\star) = Y$, the definition of a natural transformation says only that the following diagram commutes for each $g \in G$:

$$\begin{array}{ccc} X & \xrightarrow{g_*} & X \\ {\scriptstyle \eta_\star}\downarrow & & \downarrow{\scriptstyle \eta_\star} \\ Y & \xrightarrow{g_*} & Y \end{array}$$

Thus a natural transformation $\eta : \mathcal{X} \to \mathcal{Y}$ consists just of a map of sets $\eta_\star : X \to Y$ which commutes with the G-action. This is exactly a map of G-sets.

(c) Recall the functor $Gl_n : \mathbf{Comm} \to \mathbf{Gp}$ from above. We define a new functor $(-)^\times : \mathbf{Comm} \to \mathbf{Gp}$ which takes a commutative ring R and gives $R^\times$, the units (invertible elements) in R. Note that $R^\times$ is a group under multiplication and that any ring homomorphism $f : R \to S$ induces a group homomorphism $f^\times : R^\times \to S^\times$. Now the determinant yields a natural transformation $\det : Gl_n \to (-)^\times$; to see that this is natural, i.e., that each diagram

$$\begin{array}{ccc} Gl_n(R) & \xrightarrow{Gl_n(f)} & Gl_n(S) \\ {\scriptstyle \det_R}\downarrow & & \downarrow{\scriptstyle \det_S} \\ R^\times & \xrightarrow{f^\times} & S^\times \end{array}$$

commutes, we need only note that the determinant of a matrix over any ring is given by a formula not depending on the ring, so that we get the same answer whether we change coefficients before or after calculating the determinant. Also, note that $\det_R : Gl_n(R) \to R^\times$ is required to be a morphism in the category $\mathbf{Gp}$, which is the condition that the determinant of a product of matrices is the product of the determinants.

(d) Let k be a field. For any vector space V over k, we define the dual vector space

$$V^* := \operatorname{Hom}_k(V, k).$$

This is the vector space of linear functionals on V. In fact the assignment $V \mapsto V^*$ determines a contravariant functor $(-)^* : \mathbf{Vect}_k \to \mathbf{Vect}_k$. Composing this functor with itself gives a covariant functor $(-)^{**} : \mathbf{Vect}_k \to \mathbf{Vect}_k$ which sends a vector space to its double dual.
Now fix $v \in V$. We define a function $eval_v : V^* \to k$ by $eval_v(\lambda) = \lambda(v)$. This is in fact k-linear and so determines an element of $(V^*)^*$. But now the assignment $v \mapsto eval_v$ can also be seen to be k-linear, so we have a homomorphism $eval : V \to V^{**}$. This map is an isomorphism if V is finite dimensional. Moreover, the homomorphisms $V \to V^{**}$ fit together to determine a natural transformation of functors $\mathrm{Id} \to (-)^{**}$. This is a precise version of the statement that a finite-dimensional vector space is *canonically* isomorphic to its double dual.

Exercise 9. Verify that the map $eval : V \to V^{**}$ described above is natural. (First step: what does it mean for $eval$ to be natural?)

Remark. For finite-dimensional vector spaces, it is also true that V is isomorphic to V^*, but to construct such an isomorphism one must first choose a basis for V. Thus the isomorphism $V \cong V^*$ cannot be natural.

We saw that if we restrict ourselves to $(\mathbf{Vect}_k)_{\text{f.d.}}$, then $eval$ determines a natural transformation $\mathrm{Id} \to (-)^{**}$ in which each map $V \to V^{**}$ is an isomorphism. More generally, a natural transformation $\eta : F \to G$ between functors $F, G : \mathscr{C} \to \mathscr{D}$ is called a *natural isomorphism* if $\eta_C : F(C) \to G(C)$ is an isomorphism for each $C \in \mathscr{C}$.

Definition. A functor $F : \mathscr{C} \to \mathscr{D}$ is called an *equivalence* of categories if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $F \circ G \cong \mathrm{Id}_{\mathscr{D}}$ and $G \circ F \cong \mathrm{Id}_{\mathscr{C}}$.

There is also a notion of *isomorphism* of categories: a functor $F : \mathscr{C} \to \mathscr{D}$ is an isomorphism if there is a functor going in the opposite direction such that both composites are exactly the identity functor. However, this turns out to be much too strong in practice, and equivalence is a much more useful notion.

Proposition 1. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F is an equivalence if and only if F is fully faithful and for each $D \in \mathscr{D}$ there exists $C \in \mathscr{C}$ such that $F(C)$ is isomorphic to D.*

Proof. ($\Rightarrow$). Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence. Thus there exists $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta : G \circ F \to \mathrm{Id}_{\mathscr{C}}$ and $\lambda : F \circ G \to \mathrm{Id}_{\mathscr{D}}$. Now let $D \in \mathscr{D}$. Then we have an isomorphism $\lambda_D : F(G(D)) \cong D$. It thus remains to show that F is fully faithful.

Let $f, g \in \mathrm{Hom}_{\mathscr{C}}(C, C')$, and suppose $F(f) = F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(C), F(C'))$. We wish to show that $f = g$. But since $F(f) = F(g)$, it follows that $G(F(f)) = G(F(g))$. Now, by naturality, the following diagram commutes for any $h \in \mathrm{Hom}_{\mathscr{C}}(C, C')$:

$$\begin{array}{ccc} G(F(C)) & \xrightarrow{G(F(h))} & G(F(C')) \\ {\scriptstyle \eta_C}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \eta_{C'}} \\ C & \xrightarrow{\quad h \quad} & C' \end{array}$$

In particular, substituting in f and g for h, we get the same morphism at the top in the two cases. We now have

$$f = \eta_{C'} \circ G(F(f)) \circ \eta_C^{-1} = \eta_{C'} \circ G(F(g)) \circ \eta_C^{-1} = g.$$

It follows that F is faithful.

Finally, let $g \in \mathrm{Hom}_{\mathscr{D}}(F(C), F(C'))$. We want to show that $g = F(f)$ for some $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$. We define f to be

$$f = \eta_{C'} \circ G(g) \circ \eta_C^{-1}.$$

But we have a commutative diagram

$$\begin{array}{ccc} G(F(C)) & \xrightarrow{G(F(f))} & G(F(C')) \\ {\scriptstyle \eta_C}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \eta_{C'}} \\ C & \xrightarrow{\quad f \quad} & C' \end{array},$$

so that $f = \eta_{C'} \circ G(F(f)) \circ \eta_C^{-1}$. Since $\eta_{C'}$ and η_C are isomorphisms, this implies that $G(g) = G(F(f))$. But now we have already shown that any equivalence is faithful; in particular, G is faithful, so $g = F(f)$ as required. It now follows that F is full.

($\Leftarrow$). Let $F : \mathscr{C} \to \mathscr{D}$ be fully faithful and suppose that for each $D \in \mathscr{D}$ there exists $C \in \mathscr{C}$ such that $F(C)$ is isomorphic to D. Now we choose at once, for each $D \in \mathscr{D}$, an object $C_D \in \mathscr{C}$ and an isomorphism $\varphi_D : F(C_D) \xrightarrow{\cong} D$. We now define a functor $G : \mathscr{D} \to \mathscr{C}$ as follows. On objects, we set $G(D) = C_D$. Given any morphism $f : D \to D'$, we define $G(f)$ to be the unique morphism $C_D \to C_{D'}$ such that $F(G(f))$ is the composite

$$F(C_D) \xrightarrow{\varphi_D} D \xrightarrow{f} D' \xrightarrow{\varphi_{D'}} F(C_{D'}).$$

Note that this makes sense because F is fully faithful. It is easy to see that $G(1_D) = 1_{C_D}$ and that $G(g \circ f) = G(g) \circ G(f)$. Thus G is in fact a functor.

It remains to define natural isomorphisms $\eta : G \circ F \to \mathrm{Id}_{\mathscr{C}}$ and $\lambda : F \circ G \to \mathrm{Id}_{\mathscr{D}}$. Let $C \in \mathscr{C}$. Then $G(F(C)) = C_{F(C)}$ by the definition of G. We then define $\eta_C : G(F(C)) \xrightarrow{\cong} C$ to be the unique morphism such that

$$F(\eta_C) = \varphi_{F(C)} : F(C_{F(C)}) \xrightarrow{\cong} F(C).$$

We are once again using the fact that F is fully faithful; this also implies that η_C is an isomorphism since $\varphi_{F(C)}$ is one. We must still check that this choice of η is natural, i.e., that each diagram

$$\begin{array}{ccc} G(F(C)) & \xrightarrow{G(F(f))} & G(F(C')) \\ \downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_{C'}} \\ C & \xrightarrow{f} & C' \end{array}$$

commutes for each choice of C, C', and f. But since F is fully faithful, it suffices to check that the diagram commutes after applying F. By the definition of G and η, the diagram becomes

$$\begin{array}{ccc} F(C_{F(C)}) & \xrightarrow{F(G(F(g)))} & F(C_{F(C')}) \\ \downarrow{\scriptstyle \varphi_{F(C)}} & & \downarrow{\scriptstyle \varphi_{F(C')}} \\ F(C) & \xrightarrow{F(f)} & F(C') \end{array} .$$

This diagram commutes by the definition of $F(G(F(f)))$.

The natural transformation is much easier to describe. We define $\lambda_D : F(G(D)) = F(C_D) \to D$ to simply be the isomorphism φ_D. Naturality requires each diagram

$$\begin{array}{ccc} F(G(D)) & \xrightarrow{F(G(f))} & F(G(D')) \\ \downarrow{\scriptstyle \varphi_D} & & \downarrow{\scriptstyle \varphi_{D'}} \\ D & \xrightarrow{f} & D' \end{array}$$

to commute, but each such diagram commutes by the definition of $G(f)$. This completes the proof. □

Exercise 10. Let k be a field and let us define a category $\mathscr{M} = \mathbf{Matr}_k$: this category has one object for each nonnegative integer and $\mathrm{Hom}_{\mathscr{M}}(m, n) = Mat_{n\times m}(k)$, the set of $n \times m$ matrices over k. There is a functor $\mathscr{M} \to (\mathbf{Vect}_k)_{\text{f.d.}}$ sending m to the vector space k^m and sending a matrix to the corresponding linear transformation. This functor is an equivalence of categories.

4 Initial and terminal objects

In most of the categories mentioned above, there are certain objects that have special properties which are easily expressible in terms of just the other objects and the morphisms, in other words only in terms of concepts that make sense in every category. In this section we'll talk about some of these objects.

Look for instance at the category of groups. The simplest group is the trivial group **1**. One way to think of **1** is as consisting of only one element, the identity. Another way though, more in touch with the look we take on things in these notes, is to notice that for any other group G, there is always *exactly* one morphism from **1** to G, namely the one sending the identity to the identity. No other group has that property, something we'll leave to you as an exercise.

Exercise 11. Show that if a group E has the property that for any group G there is exactly one morphism $E \to G$, then E consists only of the identity element.

Let's now look at the category of rings with unit. We include in that the zero ring, in which case $0 = 1$. One very important ring is the ring of integers, $\mathbb{Z}$. Notice that $\mathbb{Z}$ has the same property: For any other ring A, there is exactly one morphism of rings with unit from $\mathbb{Z}$ to A, sending 1 in $\mathbb{Z}$ to 1 in A. Notice that the zero ring does not have this property, since 1 in the zero ring is the same as 0, and hence under a ring homomorphism would have to be mapped to both 1 and 0 in A, which of course will in general be different. Notice though, that if we had not required our ring homomorphisms to take 1 to 1, then the zero ring would have been the one with the above property.

Now we can generalize this notion:

Definition. In a category $\mathscr{C}$, an object e is called *initial* if it has the property that for any other object $A \in \mathscr{C}$ there is exactly one morphism $e \to A$.

To show the usefulness of this, we're going to prove once and for all that initial objects in a category are all isomorphic to each other, and in fact with a unique isomorphism (This last statement's importance will probably not make sense to you for another couple of years, so don't worry if it seems strange right now).

Proposition 2. *If e and e' are initial objects in a category, then there is a unique isomorphism $e \to e'$.*

Proof. Since e is an initial object, there is a (unique) morphism $f : e \to e'$. Since e' is an initial object, there is a morphism $g : e' \to e$. Then we get a morphism $gf : e \to e$. We also have the identity $1_e : e \to e$. Since there is exactly one morphism from e to another object, in particular from e to e, we must have that $gf = 1_e$. Similarly, we must have that $fg = 1_{e'}$. This means exactly that f is an isomorphism. □

Similarly, one could talk about *terminal* objects. These are objects e with the property that for any other object A there is a unique morphism $A \to e$.

Exercise 12. In all the categories above, find (if any) what the initial and terminal objects are. Don't forget to prove that they are indeed initial objects and terminal objects respectively.

As you will notice by doing the examples above, sometimes it happens that an initial object is also a terminal one. In that case, it is called a *zero* object. If a category has a zero object, then between any two objects A and B there is always a *zero morphism*, which arises simply by composing the unique morphism $A \to 0$ with the unique morphism $0 \to B$. Notice that it is not a priori obvious that two zero objects will give rise to the same zero morphism.

Exercise 13. Show that the zero morphisms corresponding to two different (but of course isomorphic) objects are indeed the same morphism. In other words, even if there might be lots of different zero objects, there is always exactly one zero morphism.

You might ask, how is it possible that there will be two zero objects? Well, think for instance of the category of vector spaces. Then there are lots of spaces with only one element. For instance, the vector space consisting of $0 \in \mathbb{R}$. Or another one is the vector space consisting of $(0,0) \in \mathbb{R}^2$. These two are of course isomorphic, but they are *not the same*. But the zero morphism between two vector spaces is unique, because it takes any vector in the source space to the zero element of the target space.

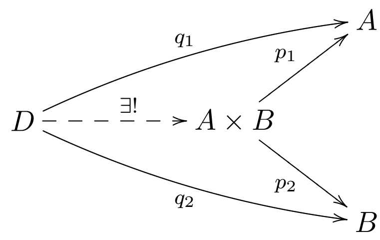
Exercise 14. Does a functor of categories have to take an initial/terminal/zero object to an initial/terminal/zero object? Prove or provide a counterexample or two.

5 Products, Kernels and other limits

One of the first and simplest accomplishments of category theory is to unify certain constructions that are similar across different categories. This is what we did above with the notions of initial, terminal and zero objects. Take a minute to ask yourselves, what other constructions are there that you've seen repeat themselves across different categories?

We are going to provide some more answers in this section. One typical construction is the following. Given two groups G, H we can construct their product $G \times H$. What properties does this product have? Well, first of all, it comes with two morphisms $p_1 : G \times H \to G$ and $p_2 : G \times H \to H$. But so do lots of other groups, for instance any subgroup of $G \times H$. In fact, there are probably lots of groups that come equipped, or that could be equipped, with morphisms to G and H. What is so special about the product? What's special about it is that it satisfies a "universal property", which makes it the best possible group with morphisms to G and H:

Definition. Given two objects A, B, an object C together with two morphisms $p_1 : C \to A$, $p_2 : C \to B$ (called the projections) is called a *product* of A and B, and is denoted by $A \times B$, if for any other object D and morphisms $q_1 : D \to A$ and $q_2 : D \to B$ there is exactly one morphism $g : D \to C$ such that $p_1 g = q_1$ and $p_2 g = q_2$.



This may sound like a very weird condition at first, but it turns out to be pretty useful, and not so weird after all. Basically it says that the object C captures exactly the information of giving morphisms to A and B. It says that giving a pair of morphisms to A and to B is exactly the same as specifying a morphism to C. It would of course be really bad if there were essentially different objects with this property.

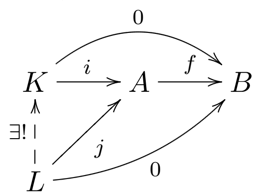
Exercise 15. Given two objects A, B, assume that C and D are both products of A and B. Show that there is a unique isomorphism $g : C \to D$ that commutes with the projections (i.e. with the notations above, $q_1 g = p_1$ and $q_2 g = p_2$).

Similarly there is the notion of a co-product, which comes with morphisms from A and B, and such that for any other object with morphisms from A and B there is a unique homomorphism from the co-product to those other objects that commutes with the injections (the morphisms from A,B).

Exercise 16. Check that in the category of sets, the disjoint union of two sets is their co-product. Examine what happens in the other examples of categories.

Notice that we could define the product or co-product of more than two, in fact even infinitely many objects. In many cases, it happens that the products and co-products (at least when there are finitely many terms involved) are isomorphic objects. We'll come back to that later.

Another important construction that appears in lots of algebraic objects is that of a kernel. Recall that a kernel of a morphism $f : A \to B$ is simply the subset of elements $a \in A$ that map to zero under f. Notice that we didn't really specify what kinds of object A and B are, and the point is that it doesn't really matter. The definition is the same. So we should probably be able to phrase it using the notions of category theory. In fact this is possible: Assume A and B are objects in a category with a zero object, and f a morphism in that category. Then a kernel K for f is an object together with a morphism $i : K \to A$, such that $fi = 0$ and for any other object L and morphism $j : L \to A$ with $fj = 0$ there exists a unique morphism $g : L \to K$ such that $ig = j$.



Exercise 17. Show that in the category of vector spaces, the familiar notion of a kernel agrees with this one.

Exercise 18. Formulate and prove the statement that two kernels for the same morphism are unique up to unique isomorphism.

Exercise 19. This is possibly a hard one, but worth thinking about. Given a category $\mathscr{C}$ with a zero object, objects A, B and a morphism $f : A \to B$, we construct a new category as follows: The objects of the new category are all morphisms $g : C \to A$ such that $fg = 0$. One could more precisely say that the objects are the pairs (C, g). A morphism from $g : C \to A$ to $h : D \to A$ is a morphism $s : C \to D$ such that $hs = g$.

1. Show that this is indeed a category.
2. Show that a kernel for f is the same thing as a terminal object in this category.
3. Explain (or at least convince yourselves) that this shows in particular that a kernel is unique up to a unique isomorphism.

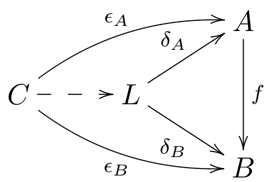
There are a number of similar constructions which we'll simply mention by name here: co-kernels, equalizers and co-equalizers, images and co-images.

In general the prefix "co-" means that to get the definition for the "co-" object you simply need to reverse the direction of all the arrow in the original definition. (So an initial object could also be called co-terminal).

Both products and kernels are special cases of a more general construction called a *limit*. This is a bit advanced, so we will only talk about it briefly.

The idea is that you start with a diagram, which just means a bunch of objects from your category, and maybe a couple of maps between them. For instance in the case of the product, the relevant diagram will be simply the two objects A, B that you want to take the product of, and no maps. In the case of the co-kernel of the morphism $f : A \to B$, the relevant diagram consists of the objects A, B and the two morphisms f and $0 : A \to B$. We'll call the diagram Δ. It is really just a subcategory of your category. Given such a diagram Δ, we say that an object L is a limit of the diagram, if it comes with morphisms $\delta_A : L \to A$, one for each object in Δ, such that:

1. For every morphism $f : A \to B$ in Δ, we have $f\delta_A = \delta_B$.
2. For every other object C together with morphisms $\epsilon_A : C \to A$ for each $A \in \Delta$, such that $f\epsilon_A = \epsilon_B$ for each $f : A \to B$ in Δ, there is a unique morphism $G : C \to L$ such that $\delta_A G = \epsilon A$ for all $A \in \Delta$.



The idea again here is that the object L, the limit, captures in the best possible way all the information of "morphisms to the objects in Δ from another object that commute with the morphisms in Δ". For completeness, note that part of the data of the limit are also the morphisms δ_A. They too are given.

Exercise 20. Try to understand why the kernel of a map is indeed the limit of the diagram described above.

One could of course talk about co-limits too. Co-kernels and co-products are examples of co-limits.

6 Universal properties and adjoint pairs

We already talked about forgetful functors and free functors in previous sections. Here, among other things, we will relate the two.

Recall that given a set X, one can construct the free abelian group on X, denoted $F(X)$. This groups has the property that it comes equipped with a set map $X \to F(X)$, and for any other abelian group G and any set map $X \to G$, there is a unique group homomorphism $F(X) \to G$ that commutes with these set maps.

Exercise 21. Show that the free group as defined earlier does indeed satisfy the above definition.

In some sense, $F(X)$ is the "best possible" abelian group corresponding to the data of the set X. Any phrase that is like the one above is called a universal property. To be more precise, starting with two categories $\mathscr{C}$ and $\mathscr{D}$, a functor $G : \mathscr{C} \to \mathscr{D}$ and an object $X \in \mathscr{D}$, we say that an object $K \in \mathscr{C}$ together with a morphism $g : X \to G(K)$ is a universal object for the pair (G, X), if for any object $L \in C$ and morphism $h : X \to G(L)$ there is a unique morphism $i : K \to L$ such that $G(i)g = h$.

This is all very abstract, so let's see what this means in the above case. There, $\mathscr{C}$ is the category of abelian groups, $\mathscr{D}$ is the category of sets, and $G = U$ is the forgetful functor, that takes a group to the underlying set. Then all the above says is that given a set X we can construct a group A, which we called $F(X)$ above, and a morphism of sets $X \to G(A)$, such that for any other group B and morphism of sets $X \to G(B)$ there is a unique morphism of groups $A \to B$, such that the corresponding morphism of sets $G(A) \to G(B)$ commutes with the maps from X. In other words, the free group on a set of generators is an example of a universal object. It is a good idea to check that you understand how the formulations above are the same.

Continuing in the same example, recall that the free group construction was actually a functor. This is in fact true more generally, in the sense that if we keep the functor G above fixed, and every object in $\mathscr{D}$ above had a universal object, then we can arrange it so that we get a functor $F : \mathscr{D} \to \mathscr{C}$. Instead of proving this in general, we will explain what it means in the case of the pair of functors (F, U).

One way to formulate the relationship between F and U above is that, given a set X and a group G, then there is a correspondence between the set maps from X to $U(G)$ and the set of group homomorphisms from $F(X)$ to G, in fact those two sets are isomorphic. The general definition is as follows:

Definition. We say that the pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ is an *adjoint pair*, if for any X in $\mathscr{C}$ and A in $\mathscr{D}$ there is an isomorphism of sets between $\mathrm{Hom}_{\mathscr{C}}(X, G(A))$ and $\mathrm{Hom}_{\mathscr{D}}(F(X), A)$ that is natural in both X and A, i.e. it behaves as one would expect when we consider a morphism $f : X \to Y$ etc. (A good exercise would be to try and make sense of that. The point is that you don't want the two sets to just be isomorphic, any two sets with the same number of elements are isomorphic. You want to somehow express the fact that there is a natural way to identify the two sets, not an ad hoc way). If the pair (F, G) is an adjoint pair, then F is called a left adjoint, and G a right adjoint.

In our (F, U) example above, the naturality discussed above is expressed in some sense by the fact that X is a subset of $F(X)$, and a homomorphism $F(X) \to G$ corresponds to its restriction $X \to U(G)$, (in other words it's the most natural map you could think of. You didn't need to make any extra assumptions.)

There are many examples of adjoint pairs of functors, so we will just mention a couple and let you work the details yourselves: (note that the pair appears in a particular order. If you consider the functors in the opposite order, it might (and probably won't) be an adjoint pair. A functor might be in two pairs of adjoint functors though, once on the left and once on the right.)

- Almost all forgetful functors have a "free" functor as left adjoint (i.e. the pair (F, U) is an adjoint pair), e.g. free group, free abelian group, free algebra, free module etc.
- There is a forgetful functor from $\mathbb{C}$-vector spaces to $\mathbb{R}$-vector spaces, forgetting extra structure. It has a left adjoint, which is extension by scalars.
- The forgetful functor from abelian groups to groups has as left adjoint the abelianization functor, which associates to each group its abelianization, i.e. the group obtained by taking the quotient with respect to the normal subgroup generated by all "commutators" $aba^{-1}b^{-1}$. (Does it have a right adjoint?)

7 Additive categories

You might have noticed that lots of examples of categories have some extra structure to their sets of morphisms that we've been ignoring. For instance, we can add morphisms of abelian groups with the same source and target. Same thing with linear transformations. One could define this in general:

Definition. An *additive category* is a category with a zero object and has products for any two objects in it, which has the extra property that all the hom-sets $\mathrm{Hom}\,(A, B)$ have the structure of an abelian group in such a way that the zero morphism is in fact the zero element of this group, and composition of functions is bi-additive.

This should not be surprising by now. Notice that we don't simply assume that the hom-sets can be equipped with a group structure, we actually equip them with one. It is part of the data.

Now, assume that X, Y are two objects in a category with a zero object, and assume that the product $X \times Y$ and the coproduct $X \amalg Y$ exist. We will now describe a natural morphism from $X \amalg Y$ to $X \times Y$. Notice that in order to get such a morphism, we would simply need to get morphisms to $X \times Y$ from X and from Y. To get a map from X to $X \times Y$, we would need to provide morphisms from X to X and to Y. But we can take the identity $1_X : X \to X$ and the zero morphism $0 : X \to Y$. These two give us a morphism $X \to X \times Y$. Similarly we get a morphism $Y \to X \times Y$, and those two together give us the desired morphism $X \amalg Y \to X \times Y$.

We will now show that in an additive category, this is actually an isomorphism. In fact, we'll try to describe its inverse. For this, let us first set up some notation. Let's call i, j the two maps from X, Y to $X \amalg Y$ respectively, and p, q the two maps from $X \times Y$ to X, Y respectively. Also, let's call $I : X \amalg Y \to X \times Y$ the morphism constructed above. Then we have the identities: $pIi = 1_X, pIj = 0, qIi = 0, qIj = 1_Y$. (Think about why we have those identities).

Now, we have two maps $ip, jq : X \times Y \to X \amalg Y$. If our category is additive, we can take their sum $J = ip + jq$. We'll show that this is the inverse to I. For that, it is enough to show that the following hold: $IJ = 1_{X \times Y}$ and $JI = 1_{X \amalg Y}$. Here we are going to use the universal properties of products and co-products. In particular, to show that $IJ = 1$, all we have to do is to show that they are equal after composing with the projections, i.e. that $pIJ = p$ and that $qIJ = q$. Now, using the bilinearity of composition, the definition of J, the identities mentioned above and the fact that the zero morphism is the zero element of the group structure, we have that $pIJ = pI(ip + jq) = pIip + pIjq = p + 0 = p$, and similarly that $qIJ = q$, so this proves the first of the two relations needed above. We leave the verification of the other relation to you.

This can be generalized to show that the product and the co-product of a *finite* set of objects are isomorphic in an additive category. Notice though that this is no longer true for infinite products and co-products.

Exercise 22. Modify the above proof to show that in an additive category, the product of two objects is indeed a co-product. (Notice that above we assumed that the co-product existed).

8 Yoneda's lemma and its importance

We described earlier the natural embedding of a vector space into its double dual, which is an isomorphism for finite-dimensional spaces. The significance of this is that the elements of the double dual are functions on some space, so what this process does is enable us to think of the elements of a vector space as functions. And functions are nice to work with.

A similar thing can be done in any category, and it goes under the name of Yoneda's lemma. To understand it a bit better, let's start with a category $\mathscr{C}$ and an object $X \in \mathscr{C}$. Then this gives rise to a functor h_X from $\mathscr{C}^{op}$ to the category of sets: To an object $Y \in \mathscr{C}^{op}$ we can associate the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ of morphisms to X, i.e. $h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$ .This is a functor. Now, we can form a category with objects all functors from $\mathscr{C}^{op}$ to the category of sets, and morphisms natural transformations. Let's denote this by $\mathscr{D}$. The above process associates to every $X \in \mathscr{C}$ an object h_X in

$\mathscr{D}$. In fact, this is a functor of categories. Yoneda's lemma basically asserts that this is a fully faithful embedding, i.e. every category can be thought of as a full subcategory of a category of functors. If none of the above makes sense, just keep in mind this: "An object can be recovered from the sets of maps from other objects to it".

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- Barry Mitchell, *Theory of Categories*, Academic Press Series in Pure and Applied Mathematics 17. Another oldie but goodie.
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