The equivariant slice spectral sequence

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Lecture 1

- Review of G-spectra
- Motivating example: $K\mathbb{R}$
- The slice filtration, $G = C_2$
- The slice spectral sequence for $K\mathbb{R}$.

Review of G-spectra (G finite)

- $Top_*^G \xrightarrow{\Sigma_G^\infty} Sp^G$ suspension G-spectrum

$$S_G^0 = \Sigma_G^\infty S^0, \quad \Sigma_G^\infty(X \wedge Y) \simeq \Sigma_G^\infty X \wedge \Sigma_G^\infty Y$$

- For $V \in Rep(G)$, $\rightsquigarrow S^V$ representation sphere

$\Sigma_G^\infty S^V$ is invertible. $\exists S^{-V}$, $S^{-V} \wedge S^V \simeq S^0$

For $X \in Sp^G$, write $\Sigma^V X = S^V \wedge X$,

$$\Sigma^{-V} X = S^{-V} \wedge X$$

Review of G-spectra (G finite)

$\mathrm{Top}_*^G \xrightarrow{\Sigma_G^\infty} \mathrm{Sp}^G$

$S_G^0 = \Sigma_G^\infty S^0$

$\Sigma_G^\infty (X \wedge Y) \simeq \Sigma_G^\infty X \wedge \Sigma_G^\infty Y$

$\Sigma^V X = S^V \wedge X,$

$\Sigma^{-V} X = S^{-V} \wedge X$

- $H \leq G$

$$\begin{tikzcd} \mathrm{Sp}^H \arrow[r, shift left, "\uparrow_H^G"] & \mathrm{Sp}^G \arrow[l, shift left, "\downarrow_H^G"] \end{tikzcd}$$

$\downarrow_H^G$ "restriction", $\uparrow_H^G = G_+ \wedge_H (-)$ "induction"

- $$\begin{tikzcd} \mathrm{Sp} \arrow[r, shift left, "q^*"] & \mathrm{Sp}^G \arrow[l, shift left, "(\ )^G"] \end{tikzcd}$$

$q^* X$ "inflation" "trivial G-action"

X^G (categorical) fixed point spectrum

- $$\begin{tikzcd} \mathrm{Sp} \arrow[rr, shift left, "G/H_+ \wedge"] \arrow[dr, shift left, "q^*"] & & \mathrm{Sp}^G \arrow[ll, shift left, "(\ )^H"] \arrow[dl, shift left, "\downarrow_H^G"] \\ & \mathrm{Sp}^H \arrow[ul, shift left, "(\ )^H"] \arrow[ur, shift left, "\uparrow_H^G"] & \end{tikzcd}$$

$X \in \mathrm{Sp}^G$

$X^H = (\downarrow_H^G X)^H$

Review of G-spectra (G finite)

⚠ **Warning** $(\Sigma^\infty_G X)^G \not\simeq \Sigma^\infty X^G$

Ex $G = C_2$ $(\Sigma^\infty_{C_2} S^0)^G \simeq \Sigma^\infty S^0 \vee \Sigma^\infty \mathbb{RP}^\infty_+$

G-space $E\mathcal{P}$, $(E\mathcal{P})^H \simeq \begin{cases} \emptyset & H = G \\ * & H \text{ proper subgroup} \end{cases}$

Ex $G = C_2$, $E\mathcal{P} = S^\infty$ antipodal action

cofiber sequence

$$E\mathcal{P}_+ \to S^0 \to \widetilde{E\mathcal{P}}$$

Define, for $X \in Sp^G$,

$$\Phi^G X = (\widetilde{E\mathcal{P}} \wedge X)^G$$

geometric fixed points

$$\begin{tikzcd} Sp^G \arrow[r, shift left, "\Phi^G"] & Sp \arrow[l, shift left, "\phi_G^*"] \end{tikzcd}$$

geometric inflation

$$\phi_G^* X = \widetilde{E\mathcal{P}} \wedge q^*(X)$$

induction

$$\begin{tikzcd} Sp^H \arrow[r, shift left, "\uparrow_H^G"] & Sp^G \arrow[l, shift left, "\downarrow_H^G"] \end{tikzcd}$$

restriction

inflation

$$\begin{tikzcd} Sp \arrow[r, shift left, "q^*"] & Sp^G \arrow[l, shift left, "(\ )^G"] \end{tikzcd}$$

G-fixed points

$$\begin{tikzcd} Sp & & Sp^G \arrow[ll, "(\ )^H"'] \arrow[dl, "\downarrow_H^G"] \\ & Sp^H \arrow[ul, "(\ )^H"] & \end{tikzcd}$$

H-fixed points

$E\mathcal{P}_+ \to S^0 \to \widetilde{E\mathcal{P}}$

$\Phi^G X = (\widetilde{E\mathcal{P}} \wedge X)^G$

geometric fixed points

$$\mathrm{Sp}^G \underset{\phi_G^*}{\overset{\Phi^G}{\rightleftarrows}} \mathrm{Sp}$$

geometric inflation

Review of G-spectra (G finite)

Properties of Geometric Fixed Points

1) $\Phi^G(\Sigma_G^\infty X) \simeq \Sigma^\infty X^G$

2) $\Phi^G(E \wedge F) \simeq \Phi^G E \wedge \Phi^G F$

$\Phi^G S^0 \simeq S^0$

$$S^0 \longrightarrow \widetilde{E\mathcal{P}}$$

induces $\quad Y^G \longrightarrow \Phi^G Y \quad$ for $\quad Y \in \mathrm{Sp}^G$

$\Phi^G(\Sigma^\infty_G X) \simeq \Sigma^\infty X^G$

$\Phi^G(E \wedge F) \simeq \Phi^G E \wedge \Phi^G F$

$\Phi^G S^0 \simeq S^0$

Review of G-spectra (G finite)

Homotopy Mackey functors $X \in Sp^G$, $H \leq G$

$\pi_n^H(X) = \pi_n(X^H)$.

If $H \leq K$

Restriction $R^K_H \; \pi_n^K(X) \longrightarrow \pi_n^H(X)$

Transfer $T^K_H \; \pi_n^H(X) \longrightarrow \pi_n^K(X)$

+ action of Weyl group $W_G(H) = \dfrac{N_G(H)}{H}$ $\circlearrowright \pi_n^H(X)$

$\Longrightarrow$ $\underline{\pi}_n(X)$ is a Mackey functor

$$\begin{tikzcd}
 & \pi_n^G(X) \arrow[dl, shift right, "R"'] \arrow[dr, shift left, "R"] & \\
W(K) \circlearrowright \pi_n^K(X) \arrow[ur, shift right, "T"'] \arrow[dr, shift right, "R"'] & & \pi_n^H(X) \circlearrowleft W(H) \arrow[ul, shift left, "T"] \arrow[dl, shift left, "R"] \\
 & \pi_n^e(X) \circlearrowleft G \arrow[ul, shift right, "T"'] \arrow[ur, shift left, "T"] &
\end{tikzcd}$$

$\pi_n(X)$ Mackey functor

$$\begin{tikzcd} & \pi_n^G(X) \arrow[dl, "R"'] \arrow[d, shift right, "R"'] \\ \pi_n^K(X) \arrow[ur, "T"'] \arrow[d, shift right, "R"'] & \pi_n^H(X) \arrow[u, shift right, "T"'] \arrow[dl, shift right, "R"'] \\ \pi_n^e(X) \arrow[u, shift right] \arrow[ur, shift right, "T"'] & \end{tikzcd}$$

$W(K) \circlearrowright \pi_n^K(X)$, $\pi_n^H(X) \circlearrowleft W(H)$, $\pi_n^e(X) \circlearrowleft G$

Review of G-spectra (G finite)

Ex $\underline{\pi}_0(S^0_G) \cong \underline{A}$ Burnside

$G = C_p$ $\quad \underline{A} = \begin{tikzcd} \mathbb{Z}\{1, C_p\} \arrow[d, shift right, "(1\ p)"'] \\ \mathbb{Z} \arrow[u, shift right, "\binom{0}{1}"'] \end{tikzcd}$

$$\mathrm{Mod}_{\mathbb{Z}[G]} \underset{Q}{\overset{F}{\rightrightarrows}} \mathrm{Mack}(G)$$

$F(M)(H) = M^H$, R = inclusion

$Q(M)(H) = M/H$, T = quotient

Ex $F(\mathbb{Z}) = \underline{\mathbb{Z}} = \begin{tikzcd} \mathbb{Z} \arrow[d, shift right, "1"'] \\ \mathbb{Z} \arrow[u, shift right, "2"'] \end{tikzcd}$ "constant"

Ex $\mathbb{Z}^\sigma$ $\quad \begin{tikzcd} 0 \arrow[d, shift right] \\ \mathbb{Z}^\sigma \arrow[u, shift right] \end{tikzcd}$

sign action $\mathbb{Z}^\sigma \circlearrowleft$

Review of G-spectra (G finite)

Eilenberg-Mac Lane G-spectra

For $\underline{M} \in Mack(G)$, $\exists\, H_G\underline{M} \in Sp^G$ w/

$$\underline{\pi}_n(H_G\underline{M}) = \begin{cases} \underline{M} & n=0 \\ 0 & \text{else} \end{cases}$$

$\underline{\pi}_*$-isomorphisms

$f: X \to Y$ in Sp^G $\rightsquigarrow$ isomorphism in $Ho(Sp^G)$ $\iff$ $f_*: \underline{\pi}_n X \to \underline{\pi}_n Y$ isomorphism in $Mack(G)$ $\forall n \in \mathbb{Z}$

$\Updownarrow$

$f^H: X^H \xrightarrow{\sim} Y^H$ in Sp $\forall H \leq G$

$\underline{\pi}_0(S^0_G) \cong \underline{A}$

$$Mod_{\mathbb{Z}[G]} \overset{F}{\underset{Q}{\rightrightarrows}} Mack(G)$$

$$\underline{\mathbb{Z}} = \begin{array}{c} \mathbb{Z} \\ 1 \downarrow \uparrow 2 \\ \mathbb{Z} \end{array}$$

$$\underline{\mathbb{Z}}^{\sigma} = \begin{array}{c} 0 \\ \\ \mathbb{Z}^{\sigma} \circlearrowleft \end{array}$$

$$f : X \xrightarrow{\sim} Y \in Sp^G$$
$$\Updownarrow$$
$$f^H : X^H \xrightarrow{\sim} Y^H \in Sp \quad \forall H \leq G$$
$$\Updownarrow$$
$$f_* : \pi_n X \xrightarrow{\cong} \pi_n Y \quad \forall n \in \mathbb{Z}$$

Atiyah's Real K-theory

- Use $C_2 \circlearrowright \mathbb{C}$ to promote $KU \in Sp \rightsquigarrow K\mathbb{R} \in Sp^{C_2}$
- $\downarrow_e^{C_2} K\mathbb{R} = KU$, $(K\mathbb{R})^{C_2} \simeq KO$.
- Bott periodicity: $\downarrow_e^{C_2} K\mathbb{R}$ 2-periodic,
  $(K\mathbb{R})^{C_2}$ 8-periodic

  $$\Rightarrow \pi_{n+8} K\mathbb{R} \cong \pi_n K\mathbb{R}$$
- $\mathbb{R}$-Bott periodicity: $\Sigma^\rho K\mathbb{R} \simeq K\mathbb{R}$
  $\rho = \mathbb{C} \circlearrowleft C_2 = \mathbb{R}[C_2]$ regular representation

$\downarrow_e^{C_2} K\mathbb{R} = KU$

$(K\mathbb{R})^{C_2} \simeq KO$

$\underline{\pi}_{n+8} K\mathbb{R} \cong \underline{\pi}_n K\mathbb{R}$

$\Sigma^{\sigma} K\mathbb{R} \simeq K\mathbb{R}$

Atiyah's Real K-theory

Problem $\mathbb{R}$-Bott periodicity of $K\mathbb{R}$ not detected in Postnikov filtration.

Response Define new filtration for C_2-spectra

- Restricts to Postnikov filtration $\downarrow_e^{C_2} : Sp^{C_2} \longrightarrow Sp$
- Interacts well with $\Sigma^{\sigma} : Sp^{C_2} \longrightarrow Sp^{C_2}$ (compatible w/ $\mathbb{R}$-Bott periodicity)

$$P_{k+2}^{n+2}(\Sigma^{\sigma} X) \simeq \Sigma^{\sigma} P_k^n(X)$$

Goal

new filtration for C_2-spectra

- $\downarrow^{C_2}_{e} P^n_k X \simeq P^n_k \downarrow^{C_2}_{e} X$
- $P^{n+2}_{k+2}(\Sigma^\rho X) \simeq \Sigma^\rho P^n_k(X)$

The (regular) slice filtration $(G = C_2)$

Want Fiber sequences

$$P_n X \longrightarrow X \longrightarrow P^{n-1} X$$

"$\geq n$" "$< n$"

and

$$P^{\ell}_{k+1} X \longrightarrow P^{\ell}_{j} X \longrightarrow P^{k}_{j} X \quad \text{for } j < k < \ell$$

e.g. $P^{k+1}_{k+1} X \longrightarrow P^{k+1}_{j} X \longrightarrow P^{k}_{j} X$

"$k+1$-slice"

Want

$P_n X \to X \to P^{n-1} X$
"$\geq n$" "$< n$"

$P^{\ell}_{k+1} X \to P^{\ell}_{j} X \to P^{k}_{j} X$

The (regular) slice filtration ($G = C_2$) HHR Ullman

Define Full subcat $\tau_{\geq n} \subseteq Sp^{C_2}$,
smallest containing
- $S^{k\rho}$, $2k \geq n$
- $\uparrow_e^{C_2} S^k$, $k \geq n$

& closed under
- isomorphisms
- wedges & cofibers (hocolims)
- extensions

Write $X \geq n$ for $X \in \tau_{\geq n}$. Say X slice n-connective

Ex $\tau_{\geq 0} = (Sp^{C_2})_{\geq 0}$ connective C_2-spectra

Ex $\tau_{\geq 1} = (Sp^{C_2})_{\geq 1}$ 1-connective C_2-spectra

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$\tau_{\geq n} \subseteq Sp^{C_2}$ generated under hocolim's, extension by $S^{k\rho}$, $2k \geq n$ & $\uparrow_e^{C_2} S^k$, $k \geq n$

If $X \in \tau_{\geq n}$, write $X \geq n$, say X slice n-connective

The (regular) slice filtration $(G = C_2)$

Define $X < n \iff [W, X] = 0 \ \forall W \in \tau_{\geq n}$

or $X \leq n-1$.

Say X slice $(n-1)$-coconnective.

Bousfield loc $\rightsquigarrow \exists\ P^n(\ ): Sp^G \to Sp^G$

s.t. $P^n(X) \leq n$ & $X \to P^n(X)$ universal.

Define • $\underbrace{P_{n+1}(X)}_{\geq n+1} = \mathrm{fib}(X \to \underbrace{P^n(X)}_{\leq n})$.

• $P^n_k(X) = P_k P^n X$

$X < n$

$\Updownarrow$

$[W, X] = 0$

$\forall\, W \in \tau_{\geq n}$

X slice $(n-1)$-coconnective

$X \to P^n(X)$ $\leq n$

universal

$\underset{\geq n+1}{P_{n+1}(X)} = \mathrm{fib}(X \to \underset{\leq n}{P^n(X)})$

$P^n_k(X) = P_k P^n X$

The (regular) slice filtration ($G = C_2$)

$\boxed{\text{Properties}}$

(HHR)

- $P^0_0 X \simeq H\underline{\pi}_0 X$
- $P^{n+2}_{k+2}(\Sigma^\rho X) \simeq \Sigma^\rho P^n_k(X)$
- $P^1_1 X \simeq \Sigma^1 H\, \underline{\pi}_1 X / \ker R$
- If $\underset{\geq n+1}{X} \to Y \to \underset{\leq n}{Z}$ fiber seqn

  then $X \simeq P_{n+1} Y$, $Z \simeq P^n Y$

If it <u>looks</u> like a slice tower, it <u>is</u> a slice tower.

$\downarrow_e^{C_2} K\mathbb{R} = KU$

$(K\mathbb{R})^{C_2} \simeq KO$

$\boxed{\text{Properties}}$

$P_0^0 X \simeq H\underline{\pi}_0 X$

$P_{k+2}^{n+2}(\Sigma^\rho X) \simeq \Sigma^\rho P_k^n(X)$

$P_1^1 X \simeq \Sigma^1 H \, \underline{\pi}_1 X / \ker R$

$$\begin{array}{ccccc} \geq n+1 & & & & \leq n \\ P_{n+1} Y & \to & Y & \to & P^n Y \\ \text{IS} & & \| & & \text{IS} \\ P_{n+1} Y & \to & Y & \to & P^n Y \end{array}$$

Looks like slice
$\Rightarrow$ is slice

Atiyah's Real K-theory

- $$\begin{tikzcd} (K\mathbb{R})^{C_2} \arrow[d, bend right] \\ \downarrow_e^{C_2} K\mathbb{R} \arrow[u, bend right] \end{tikzcd} \qquad \begin{tikzcd} KO \arrow[d, bend right, "c"'] \\ KU \arrow[u, bend right, "r"'] \end{tikzcd} \overset{\pi_0}{\rightsquigarrow} \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd} = \underline{\mathbb{Z}} = \underline{\pi}_0 K\mathbb{R}$$

  constant Mackey functor

$$\Rightarrow P_0^0 K\mathbb{R} \simeq H_{C_2} \underline{\mathbb{Z}}$$

- $$\begin{tikzcd} KO \arrow[d, bend right, "c"'] \\ KU \arrow[u, bend right, "r"'] \end{tikzcd} \overset{\pi_1}{\rightsquigarrow} \begin{array}{c} \mathbb{Z}/2 \\ \\ 0 \end{array} = \underline{g} \quad \text{geometric} \quad B(1,0)$$

$$\Rightarrow P_1^1 K\mathbb{R} \simeq \Sigma^1 H_{C_2} \underline{g}/\ker R \simeq *$$

Atiyah's Real K-theory

Dugger

HHR C_4

$\Sigma^{\rho} K\mathbb{R} \simeq K\mathbb{R}$

$P_0^0 k\mathbb{R} \simeq H_{C_2}\underline{\mathbb{Z}}$

$P_1^1 k\mathbb{R} \simeq *$

$P_{n+2}^{n+2} \Sigma^{\rho} X \simeq \Sigma^{\rho} P_n^n X$

- $$P_2^2 K\mathbb{R} \simeq \Sigma^{\rho} P_0^0 \Sigma^{-\rho} K\mathbb{R} \simeq \Sigma^{\rho} P_0^0 K\mathbb{R} \simeq \Sigma^{\rho} H\underline{\mathbb{Z}}$$

- $$P_3^3 K\mathbb{R} \simeq \Sigma^{\rho} P_1^1 \Sigma^{-\rho} K\mathbb{R} \simeq \Sigma^{\rho} P_1^1 K\mathbb{R} \simeq *$$

- $$P_n^n K\mathbb{R} \simeq \begin{cases} \Sigma^{\frac{n}{2}\rho} H\underline{\mathbb{Z}} & n \text{ even} \\ * & n \text{ odd} \end{cases}$$

Atiyah's Real K-theory

$P^{2n}_{2n} K\mathbb{R} \simeq \Sigma^{n\delta} H\mathbb{Z}$

$P^{2n+1}_{2n+1} K\mathbb{R} \simeq *$

- $k\mathbb{R} \to K\mathbb{R}$ connective cover

  $\downarrow^{C_2}_{e} k\mathbb{R} \simeq ku$, $(k\mathbb{R})^{C_2} \simeq ko$

- connective cover $= P_0$

$\Rightarrow$ $$P^n_n k\mathbb{R} \simeq \begin{cases} \Sigma^{\frac{n}{2}\delta} H\underline{\mathbb{Z}}, & n \text{ even} \geq 0 \\ * & \text{else} \end{cases}$$

$P^{2n}_{2n} K\mathbb{R} \simeq \Sigma^{n\rho} H\underline{\mathbb{Z}}$

$P^{2n+1}_{2n+1} K\mathbb{R} \simeq *$

$P^{2n}_{2n} k\mathbb{R} \simeq \Sigma^{n\rho} H\underline{\mathbb{Z}}$ $n \geq 0$

$P^{2n+1}_{2n+1} k\mathbb{R} \simeq *$

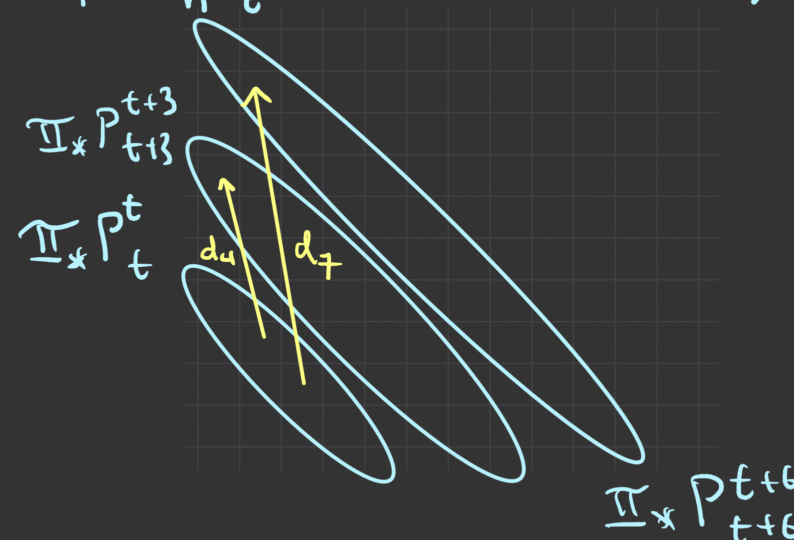
$P^{-1} k\mathbb{R} \simeq *$

Atiyah's Real K-theory

The slice spectral sequence

$$E_2^{s,t} = \underline{\pi}_{t-s} P^t_t X \Longrightarrow \underline{\pi}_{t-s} X.$$

$$d_r : \underline{\pi}_n P^t_t X \longrightarrow \underline{\pi}_{n-1} P^{t+r-1}_{t+r-1} X$$



$$\underline{\pi}_{n-1} P^{t+r-1}_{t+r-1}$$

$$\uparrow d_r$$

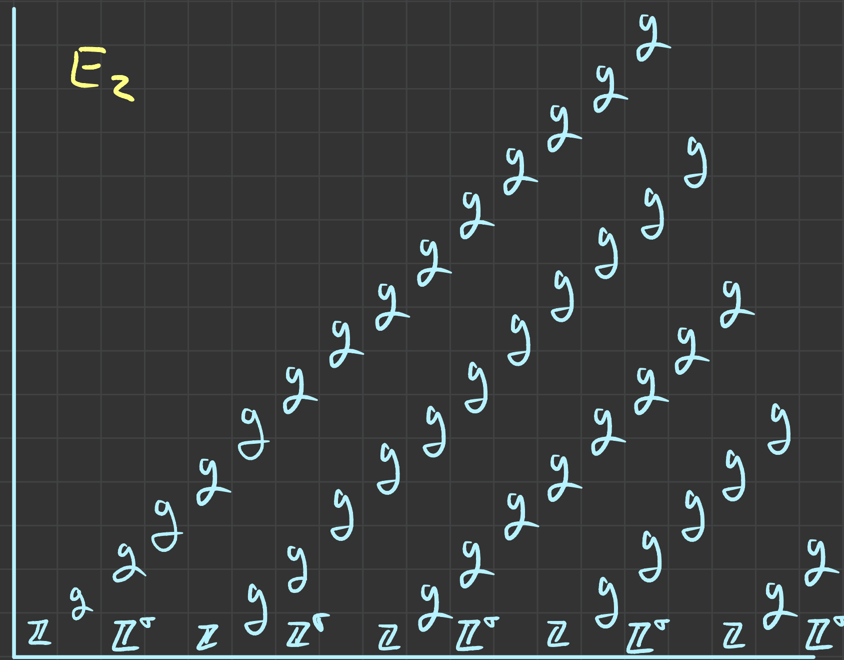
$$\underline{\pi}_n P^t_t$$

$\downarrow^{C_2}_e k\mathbb{R} \simeq ku$,

$(k\mathbb{R})^{C_2} \simeq ko$

Atiyah's Real K-theory

The slice ss for $k\mathbb{R}$



$$\underline{\mathbb{Z}} = \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$$

$$\underline{\mathbb{Z}}^\sigma = \begin{tikzcd} 0 \\ \mathbb{Z}^\sigma \end{tikzcd} \circlearrowleft \text{sign}$$

$$\underline{g} = \begin{tikzcd} \mathbb{Z}/2 \\ 0 \end{tikzcd}$$

Atiyah's Real K-theory

$$\underline{\pi}_{n-1} P^{t+r-1}_{t+r-1}$$

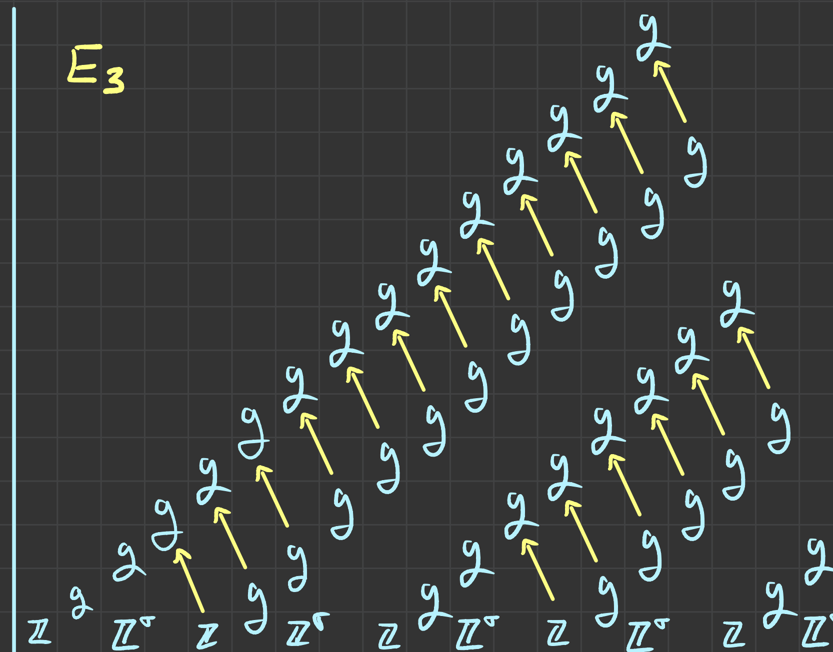
$$\Big\uparrow d_r$$

$$\underline{\pi}_n P^t_t$$

$\downarrow_e^{C_2} k\mathbb{R} \simeq ku$,

$(k\mathbb{R})^{C_2} \simeq ko$

The slice ss for $k\mathbb{R}$



$$\underline{\mathbb{Z}} = \begin{array}{c} \mathbb{Z} \\ {\scriptstyle 1}\big\downarrow \big\uparrow{\scriptstyle 2} \\ \mathbb{Z} \end{array}$$

$$\underline{\mathbb{Z}}^\sigma = \begin{array}{c} 0 \\ \\ \mathbb{Z}^\sigma \circlearrowleft \text{sign} \end{array}$$

$$\underline{g} = \begin{array}{c} \mathbb{Z}/2 \\ \\ 0 \end{array}$$

$$\underline{\pi}_n P^t_t \xrightarrow{d_r} \underline{\pi}_{n-1} P^{t+r-1}_{t+r-1}$$

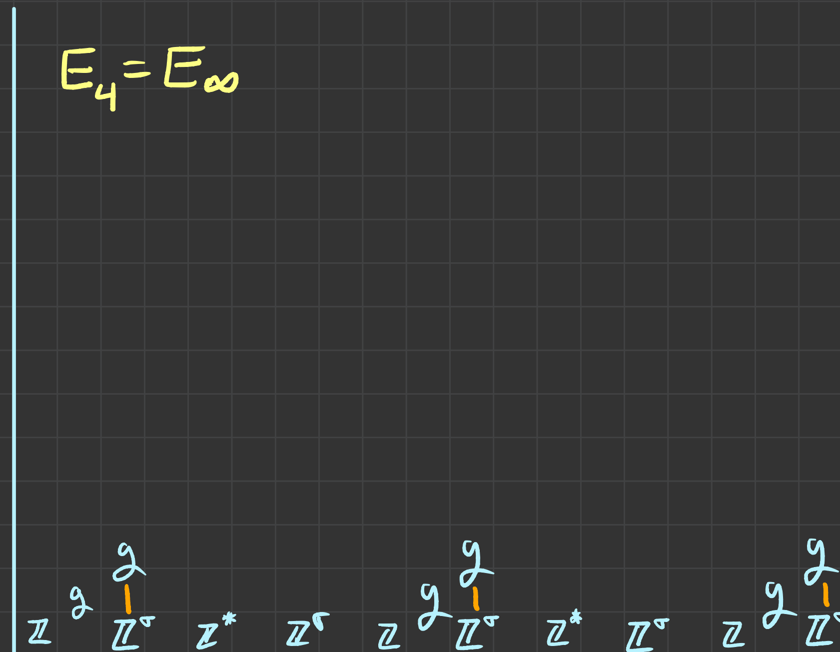
$\downarrow^{C_2}_e k\mathbb{R} \simeq ku$,

$(k\mathbb{R})^{C_2} \simeq ko$

Atiyah's Real K-theory

The slice ss for $k\mathbb{R}$

$E_4 = E_\infty$



$$\underline{\mathbb{Z}} = \begin{array}{c}\mathbb{Z}\\ 1\big\downarrow\big\uparrow 2\\ \mathbb{Z}\end{array}$$

$$\underline{\mathbb{Z}}^* = \begin{array}{c}\mathbb{Z}\\ 2\big\downarrow\big\uparrow 1\\ \mathbb{Z}\end{array}$$

$$\underline{\mathbb{Z}}^\sigma = \begin{array}{c}0\\ \\ \mathbb{Z}^\sigma\end{array}$$

$$\underline{g} = \begin{array}{c}\mathbb{Z}/2\\ \\ 0\end{array}$$

Lecture 2

- Bredon homology ~~/~~
- The slice filtration, general G
- Slice filtration for $\Sigma^{V} H_{C_p}\underline{\mathbb{Z}}$