The equivariant slice spectral sequence

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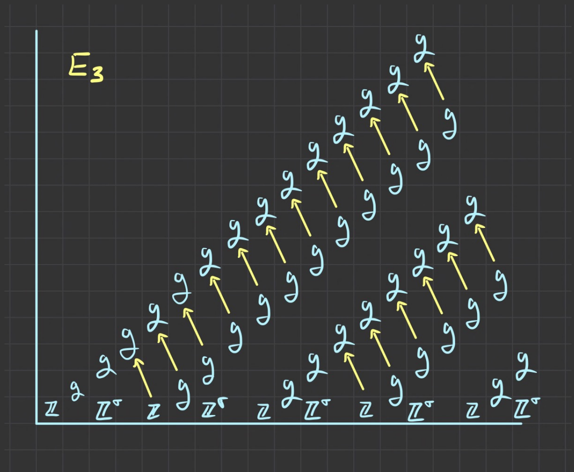
Lecture 1

- Review of G-spectra
- Motivating example: $K\mathbb{R}$
- The slice filtration, $G = C_2$

$$P^n_n K\mathbb{R} \simeq \begin{cases} \Sigma^{\frac{n}{2}\rho} H\underline{\mathbb{Z}} & n \text{ even} \\ * & n \text{ odd} \end{cases}$$

- The slice spectral sequence for $K\mathbb{R}$.



Lecture 2

- Bredon homology
- The slice filtration, general G
- Examples: $\Sigma^V H_G \underline{\mathbb{Z}}$

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Bredon Homology

Last time: $\underline{\pi}_n P_t^t k\mathbb{R} \Rightarrow \underline{\pi}_n k\mathbb{R}$

$$\underline{\pi}_n P_t^t k\mathbb{R} = \underline{\pi}_n \Sigma^{\frac{t}{2}\rho} H\underline{\mathbb{Z}} \; ?$$

Bredon homology $X \in \mathrm{Top}_*^G$, $\underline{M} \in \mathrm{Mack}(G)$

$$\tilde{H}_n(X;\underline{M}) = \pi_n^G(X \wedge H_G\underline{M})$$

$$\underline{\tilde{H}}_n(X;\underline{M}) = \underline{\pi}_n(X \wedge H_G\underline{M})$$

(Clover May's 2020 Minicourse)

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$\tilde{H}_n(X;\underline{M}) = \pi_n^G(X \wedge H_G\underline{M})$

$\underline{\tilde{H}}_n(X;\underline{M}) = \underline{\pi}_n(X \wedge H_G\underline{M})$

Bredon Homology (Clover May's 2020 Minicourse)

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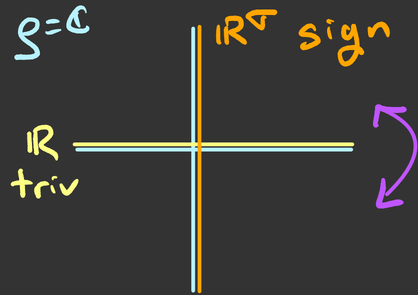
Last time ($G=C_2$)

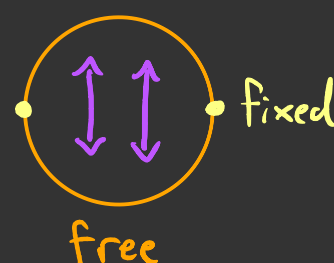
$$\underline{\pi}_n(\Sigma^\rho H_{C_2}\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^\sigma & n=2 \\ \underline{g} & n=1 \\ 0 & \text{else} \end{cases}$$

$\rho = \mathbb{C}$



$\rho \cong 1 \oplus \sigma$, $S^\rho \cong \Sigma^1 S^\sigma$

G-CW structure on S^σ



$C_{2+} \wedge S^0 \to S^0 \to S^\sigma$

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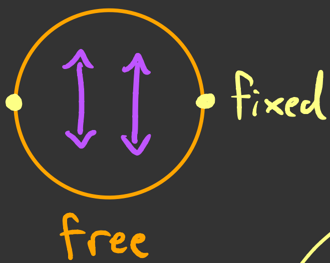
⑤

Bredon Homology (Clover May's 2020 Minicourse)



$C_{2+} \to S^0 \to S^\sigma$

Smash w/ $H\underline{\mathbb{Z}}$

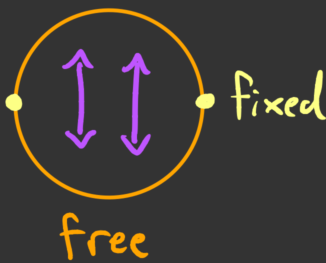
$$C_{2+} \wedge H\underline{\mathbb{Z}} \longrightarrow H\underline{\mathbb{Z}} \longrightarrow S^\sigma \wedge H\underline{\mathbb{Z}}$$

Lemma 1) $C_{2+} \wedge X \simeq \uparrow_e^{C_2} \downarrow_e^{C_2} X$

2) $\underline{\pi}_n(C_{2+} \wedge X) \cong \uparrow_e^{C_2} \pi_n(\downarrow_e^{C_2} X)$

induced Mackey functor

$M \in Ab$, $\uparrow_e^{C_2} M = $
$$\begin{tikzcd} M \arrow[d, bend right, "1+\gamma"'] \\ \mathbb{Z}[C_2] \otimes_{\mathbb{Z}} M \arrow[u, bend right] \arrow[loop right] \end{tikzcd}$$



$$\underline{\pi}_n(C_{2+}\wedge X) \cong \uparrow_e^{C_2} \pi_n^e(X)$$

$$\uparrow_e^{C_2} M = \begin{tikzcd} M \arrow[d, bend right, "1+\gamma"'] \\ \mathbb{Z}[C_2]\otimes_{\mathbb{Z}} M \arrow[u, bend right] \end{tikzcd}$$

Bredon Homology

$$C_{2+}\wedge H\underline{\mathbb{Z}} \longrightarrow H\underline{\mathbb{Z}} \longrightarrow S^\sigma \wedge H\underline{\mathbb{Z}}$$

$\underline{\pi}_*$ ⇓

$$0 \to 0 \to \underline{\tilde{H}}_1(S^\sigma;\mathbb{Z}) \to \uparrow_e^{C_2}\mathbb{Z} \to \underline{\mathbb{Z}} \to \underline{\tilde{H}}_0(S^\sigma;\mathbb{Z})$$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d, bend right, "1+\gamma"'] & \mathbb{Z} \arrow[r] \arrow[d, bend right, "1"'] & \mathbb{Z}/2 \\ \mathbb{Z}\{1-\gamma\} \arrow[r] & \mathbb{Z}[C_2] \arrow[r] \arrow[u, bend right] & \mathbb{Z} \arrow[r] \arrow[u, bend right, "2"'] & 0 \end{tikzcd}$$

$\mathbb{Z}^\sigma$ "

$$\boxed{\begin{array}{l} \underline{\pi}_1 \Sigma^\sigma H\underline{\mathbb{Z}} \cong \mathbb{Z}^\sigma \\ \underline{\pi}_0 \Sigma^\sigma H\underline{\mathbb{Z}} \cong g \end{array}}$$

Homework Compute $\underline{\pi}_n \Sigma^{k\sigma} H_{C_2}\underline{\mathbb{Z}}$

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$\underline{\pi}_n(C_{2+}\wedge X) \cong \uparrow_e^{C_2} \pi_n^e(X)$

$$\uparrow_e^{C_2} M = \begin{tikzcd} M \arrow[d, bend right, "1+\gamma"'] \\ \mathbb{Z}[C_2]\otimes_{\mathbb{Z}} M \arrow[u, bend right] \end{tikzcd}$$

$$\begin{tikzcd} C_{2+}\wedge H\underline{\mathbb{Z}} \arrow[r] & H\underline{\mathbb{Z}} \arrow[d] \\ & S^\sigma \wedge H\underline{\mathbb{Z}} \end{tikzcd}$$

$\underline{\pi}_0 \Sigma^\sigma H_{C_2}\underline{\mathbb{Z}} \cong \underline{g}$

$\underline{\pi}_1 \Sigma^\sigma H_{C_2}\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^\sigma$

Bredon Homology

$\Sigma^{-\sigma} H_{C_2}\underline{\mathbb{Z}}$

$$C_{2+} \to S^0 \to S^\sigma \overset{D(\;)}{\rightsquigarrow} S^{-\sigma} \to S^0 \to C_{2+}$$

$$D(G/H_+) \simeq G/H_+ \text{ in } Sp^G$$

$$\Sigma^{-\sigma} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \to C_{2+}\wedge H\underline{\mathbb{Z}}$$

$\underline{\pi}_*$

$$\underline{\pi}_0 \Sigma^{-\sigma} H\underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to \uparrow_e^{C_2}\mathbb{Z} \to \underline{\pi}_{-1}\Sigma^{-\sigma}\underline{\mathbb{Z}}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[r] \arrow[d, bend right, "1+\gamma"'] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "1+\gamma"] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_2] \arrow[r] \arrow[u, bend right] & \mathbb{Z}[C_2]/\mathbb{Z} \cong \mathbb{Z}^\sigma \end{tikzcd}$$

$$\boxed{\Sigma^{-\sigma} H\underline{\mathbb{Z}} \simeq \Sigma^{-1} H\underline{\mathbb{Z}}^\sigma}$$

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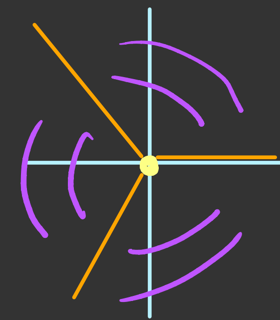
$\underline{\pi}_n(C_{2+}\wedge X) \cong \uparrow_e^{C_2} \pi_n^e(X)$

$\uparrow_e^{C_2} M = \begin{tikzcd} M \arrow[d, bend right, "1+\gamma"'] \\ \mathbb{Z}[C_2]\otimes_{\mathbb{Z}} M \arrow[u, bend right] \end{tikzcd}$

Bredon Homology $\Sigma^\rho H_{C_3}\underline{\mathbb{Z}}$

$\boxed{G=C_3}$ $\rho_{C_3} \cong 1 \oplus \lambda$

λ 2-dim rotation representation



CW structure on S^λ

$C_{3+} \to S^0 \to S^{\top} = S(C_3)$

$C_{3+}\wedge S^1 \to S^{\top} \to S^\lambda$

	0	1	2
$\underline{\pi}_n\, S^{\top}\wedge H\underline{\mathbb{Z}}$	$\underline{g}_3$	$\underline{I}$	
$\underline{\pi}_n\, S^{\lambda}\wedge H\underline{\mathbb{Z}}$	$\underline{g}_3$	0	$\underline{\mathbb{Z}}$

B(1,0)

$G = C_3$

$\rho_{C_3} \cong \underline{1} \oplus \lambda$

$S^\lambda = S^0 \cup C_{3+} \wedge e^1$
$\cup\, C_{3+} \wedge e^2$

$\underline{\pi}_0 \Sigma^\lambda H\underline{\mathbb{Z}}_{C_3} \cong \underline{g}$

$\underline{\pi}_1 \Sigma^\lambda H\underline{\mathbb{Z}}_{C_3} \cong \underline{\mathbb{Z}}$

Bredon Homology

$\boxed{G = C_4}$ $\quad \rho_{C_4} \cong \underline{1} \oplus \sigma \oplus \lambda$

σ sign representation of $C_4/C_2 \cong C_2$

λ 2-dim rotation representation

Alternative approach to $\tilde{H}_*(S^\lambda; \underline{\mathbb{Z}})$

$$S(\lambda)_+ \longrightarrow S^0 \longrightarrow S^\lambda$$

unit sphere in λ

$$S(\lambda) = S^1_{rot} = C_4 \cup C_4 \times e^1$$

$$\underline{C}_*(S(\lambda)) = \uparrow_e^{C_4} \underline{\mathbb{Z}} \xrightarrow{1-\gamma} \uparrow_e^{C_4} \underline{\mathbb{Z}}$$

$$\begin{array}{ccccccccc}
 & \mathbb{Z} & \to & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to & \mathbb{Z} & \\
 & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle 1+\gamma}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & \\
\underline{\mathbb{Z}} & \mathbb{Z} & \to & \mathbb{Z}[C_4/C_2] & \xrightarrow{1-\gamma} & \mathbb{Z}[C_4/C_2] & \to & \mathbb{Z} & \underline{\mathbb{Z}}^* \text{ dual constant} \\
 & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma^2}\downarrow\uparrow & & {\scriptstyle 1+\gamma^2}\downarrow\uparrow & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & \\
 & \mathbb{Z} & \to & \mathbb{Z}[C_4] & \xrightarrow{1-\gamma} & \mathbb{Z}[C_4] & \to & \mathbb{Z} &
\end{array}$$

$G = C_4$

$\rho_{C_4} \cong 1 \oplus \sigma \oplus \lambda$

$S(\lambda)_+ \longrightarrow S^0 \longrightarrow S^\lambda$

$\underline{H}_n(S(\lambda)) \cong \begin{cases} \underline{\mathbb{Z}} & n=1 \\ \underline{\mathbb{Z}}^* & n=0 \end{cases}$

Bredon Homology

$$S(\lambda)_+ \wedge H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \to S^\lambda \wedge H\underline{\mathbb{Z}}$$

$$\begin{array}{llll} n=2 & & & \underline{\mathbb{Z}} \\ n=1 & \underline{\mathbb{Z}} & \cong & \\ n=0 & \underline{\mathbb{Z}}^* \xrightarrow{\ 1\ } & \underline{\mathbb{Z}} & \longrightarrow B(2,0) \end{array}$$

$\mathbb{Z}/4$

$\mathbb{Z}/2$

0

$$\Sigma^2 H_{C_4}\underline{\mathbb{Z}} \to \Sigma^\lambda H_{C_4}\underline{\mathbb{Z}} \to H_{C_4}\underline{B}(2,0)$$

[HHR C_4, Figure 6]

$\Sigma^\sigma \downarrow$

$$\Sigma^{2+\sigma} H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\lambda+\sigma} H_{C_4}\underline{\mathbb{Z}} \to \Sigma^\sigma H_{C_4}\underline{B}(2,0)$$

[HHR C_4, Figure 3]

$G = C_4$

$\rho_{C_4} \cong 1 \oplus \sigma \oplus \lambda$

$S^\lambda = S^0 \cup C_{4+} \wedge e^1 \cup C_{4+} \wedge e^2$

$\underline{\pi}_1 \Sigma^\lambda H_{C_4}\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$

$\underline{\pi}_0 \Sigma^\lambda H_{C_4}\underline{\mathbb{Z}} \cong \underline{B}(2,0)$

Bredon Homology

$$\Sigma^{2+\sigma} H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\lambda+\sigma} H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\sigma} H_{C_4}\underline{B}(2,0)$$

$$\begin{tikzcd}
\underline{M}(C_4) \arrow[d, shift right] \\
\underline{M}(C_2) \arrow[u, shift right] \arrow[d, shift right] \arrow[loop right, "C_2"] \\
\underline{M}(e) \arrow[u, shift right] \arrow[loop right, "C_4"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mack}(C_4/C_2) & W_{C_4}(C_2) \arrow[l, bend right] \arrow[r, bend left] & Sp^{C_4/C_2} \\
\mathrm{Mack}(C_4) \arrow[u, "q_*"] \arrow[dr, "\downarrow^{C_4}_{C_2}"'] & & Sp^{C_4} \arrow[u, "(\,)^{C_2}"] \arrow[d, "\downarrow^{C_4}_{C_2}"] \\
& \mathrm{Mack}(C_2) & Sp^{C_2}
\end{tikzcd}$$

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Bredon Homology

$$\begin{array}{cc} \mathrm{Mack}(C_4/C_2) & Sp^{C_4/C_2} \\ q_* \uparrow & ()^{C_2} \uparrow \\ \mathrm{Mack}(C_4) & Sp^{C_4} \\ \downarrow^{C_4}_{C_2} \downarrow & \downarrow^{C_4}_{C_2} \downarrow \\ \mathrm{Mack}(C_2) & Sp^{C_2} \end{array}$$

$$\Sigma^{2+\sigma} H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\lambda+\sigma} H_{C_4}\underline{\mathbb{Z}}$$
$$\downarrow$$
$$\Sigma^{\sigma} H_{C_4} \underline{B(2,0)}$$

- $\boxed{\text{Projection Formula}}$ $\quad Sp^{G/N} \underset{(\ )^N}{\overset{q^*}{\rightleftarrows}} Sp^G$ $\quad X \in Sp^{G/N}$, $Y \in Sp^G$

$$(q^* X \wedge Y)^N \simeq X \wedge Y^N \text{ in } Sp^{G/N}$$

$$\Rightarrow (S^\sigma \wedge H_{C_4}\underline{\mathbb{Z}})^{C_2} \simeq S^\sigma \wedge H_{C_2}\underline{\mathbb{Z}} \rightsquigarrow \underline{\pi}_1 \cong \underline{\mathbb{Z}}^\sigma, \rightsquigarrow \underline{\pi}_0 \cong \underline{g}$$

- $\downarrow^{C_4}_{C_2}(\Sigma^\sigma H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^1 H_{C_2}\underline{\mathbb{Z}}$

$$\Rightarrow \underline{\pi}_0(\Sigma^\sigma H_{C_4}\underline{\mathbb{Z}}) \cong \underline{g} = \begin{array}{c} \mathbb{Z}/2 \\ 0 \\ 0 \end{array} = B(2,1), \quad \underline{\pi}_1(\Sigma^\sigma H_{C_4}\underline{\mathbb{Z}}) \cong \begin{array}{c} 0 \\ \mathbb{Z}^\sigma \\ \downarrow \uparrow 2 \\ \mathbb{Z}^\sigma \end{array}$$

Projection Formula

$$Sp^{G/N} \underset{(\ )^N}{\overset{q^*}{\rightleftarrows}} Sp^G$$

$X \in Sp^{G/N}$

$Y \in Sp^G$

$$(q^* X \wedge Y)^N \simeq X \wedge Y^N$$

in $Sp^{G/N}$

$$\underline{\pi}_1 \Sigma^\sigma H\underline{\mathbb{Z}}_{C_4} \cong \underline{\mathbb{Z}}^\sigma$$

$$\underline{\pi}_0 \Sigma^\sigma H\underline{\mathbb{Z}}_{C_4} \cong g = \underline{B}(2,1)$$

The (regular) slice filtration

Define Full subcat $\tau_{\geq n} \subseteq Sp^G$,

smallest containing $\boxed{\uparrow_H^G S^{k\cdot\rho_H},\ \forall\ H \leq G \ \&\ k \geq 0,\ k\cdot|H| \geq n}$

& closed under
- isomorphisms
- wedges & cofibers (hocolims)
- extensions

Ex $\tau_{\geq 0} = (Sp^G)_{\geq 0}$ connective G-spectra

Define $X \leq n-1 \iff [W, X] = 0 \ \forall\ W \in \tau_{\geq n}$

$\tau_{\geq n}$ gen under hocolims & extensions by

$$\uparrow_H^G S^{k\rho_H}$$

$H \leq G \quad k \geq 0$

$k \cdot |H| \geq n$

$X \leq n-1 \iff$

$[W, X] = 0$

$\forall W \in \tau_{\geq n}$

The (regular) slice filtration $\quad G = |G|$

$\boxed{\text{Properties}}$

- $P_0^0 X \simeq H_{\underline{\pi}_0} X$
- $P_1^1 X \simeq \Sigma^1 H_{\underline{\pi}_1} X / \ker R$
- $P_{k+G}^{n+G}(\Sigma^\rho X) \simeq \Sigma^\rho P_k^n(X)$
- $P_k^n(\downarrow_H^G X) \simeq \downarrow_H^G P_k^n(X)$

[Hill] • $\Phi_G^* P_k^n X \simeq P_{kG}^{nG} \Phi_G^* X$,

only slices in multiples of G

Ex $G = C_2$, $\Sigma^1 H_{C_2} \underline{g} = \Phi_{C_2}^* \Sigma^1 H\mathbb{F}_2$ a 2-slice

$P^0_0 X \simeq H\underline{\pi}_0 X$

$G = |G|$

$P^{n+G}_{k+G}(\Sigma^\rho X) \simeq \Sigma^\rho P^n_k(X)$

$\Phi^*_G P^n_k X \simeq P^{nG}_{kG} \Phi^*_G X$

The (regular) slice filtration

$G = |G|$, $N = |N|$

$\boxed{\text{Slices \& geometric inflation}}$

$N \trianglelefteq G$

$(E\mathcal{F}[N])^H \simeq \begin{cases} \emptyset & H \geq N \\ * & \text{else} \end{cases}$

$E\mathcal{F}[N]_+ \to S^0 \to \widetilde{E\mathcal{F}[N]}$

$$Sp^G \underset{\Phi^*_N}{\overset{\Phi^N}{\rightleftarrows}} Sp^{G/N}$$

$\Phi^N(X) = (\widetilde{E\mathcal{F}[N]} \wedge X)^N$

$\Phi^*_N(Z) = \widetilde{E\mathcal{F}[N]} \wedge q^* Z$

[Hill, Ullman]

- $\Phi^*_N P^n_k X \simeq P^{nN}_{kN} \Phi^*_N X$,

only slices in multiples of N

$G = |G|$

$$\Phi_G^* P^n_k X \simeq P^{nG}_{kG}\, \Phi_G^* X$$

$N = |N|$

$$\Phi_N^* P^n_k X \simeq P^{nN}_{kN}\, \Phi_N^* X$$

The (regular) slice filtration

Ex $G = C_4$, $N = C_2$

- $\underset{\text{1-slice}}{\Sigma^1 H_{C_2}\underline{\mathbb{Z}}} \Rightarrow \underset{\text{2-slice}}{\Phi_{C_2}^* \Sigma^1 H_{C_2}\underline{\mathbb{Z}}} = \Sigma^1 H_{C_4} \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \\ 0 \end{matrix}$; 2

- $G = C_4$, $N = C_4$

  $\underset{\text{1-slice}}{\Sigma^1 H\underline{\mathbb{Z}}} \Rightarrow \underset{\text{4-slice}}{\Phi_{C_4}^* \Sigma^1 H\underline{\mathbb{Z}}} = \Sigma^1 H_{C_4} \begin{matrix} \mathbb{Z} \\ 0 \\ 0 \end{matrix}$

Lecture 3

- Examples: $\Sigma^V H_G \underline{\mathbb{Z}}$ (continued)
- The $RO(G)$-grading
- Duality