

The equivariant slice  
spectral sequence

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## Lecture 2

$$\sum^\sigma H_{C_2} \underline{\mathbb{Z}}$$

- Bredon homology  $\sum^\lambda H_{C_\lambda} \underline{\mathbb{Z}}$
- The slice filtration, general  $G$
- Examples:  $\sum^\vee H_G \underline{\mathbb{Z}}$

## Lecture 3

- Examples:  $\Sigma^v H_G \underline{\mathbb{Z}}$
- The RO(G)-grading
- Duality

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## Examples of slice towers $\sum^7 H_{C_2} \underline{\mathbb{Z}}$

Homework  $\sum^n H_{C_2} \underline{\mathbb{Z}}$  n-slice for  $0 \leq n \leq 6$

Ex  $\sum^7 H_{C_2} \underline{\mathbb{Z}}$  Is this a 7-slice?

$\sum^7 H_{C_2} \underline{\mathbb{Z}} \stackrel{?}{\simeq} \sum^{3g+1} H_{C_2} \underline{M}$ , some M w/  
 $\downarrow \sum^{-3g}$  injective  $R$

$\sum^{7-3g} H_{C_2} \underline{\mathbb{Z}} \simeq \sum^1 H_{C_2} Q \underline{\mathbb{Z}}^\sigma$  Is this a 1-slice?  
 NO

$g \hookrightarrow Q \underline{\mathbb{Z}}^\sigma \rightarrow \underline{\mathbb{Z}}^\sigma$  ②  $\sum^1 H g \rightarrow \sum^1 H Q \underline{\mathbb{Z}}^\sigma \rightarrow \sum^1 H \underline{\mathbb{Z}}^\sigma$  ①

⑧  $\sum^{1+3g} H g \rightarrow \sum^7 H \underline{\mathbb{Z}} \rightarrow \sum^{1+3g} H \underline{\mathbb{Z}}^\sigma$   
 $\sum^4 H g$  ⑦

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$$\textcircled{8} \quad \sum^u Hg \rightarrow \sum^7 H\underline{\underline{Z}}$$

$$\textcircled{7} \quad \sum^{3+2g} H\underline{\underline{Z}}^g$$

### Examples of slice towers

$$\textcircled{8} \quad \begin{matrix} \sum^{1+3g} Hg \\ \text{``} \\ \sum^u Hg \end{matrix} \rightarrow \sum^7 H\underline{\underline{Z}} \rightarrow \begin{matrix} \sum^{1+3g} H\underline{\underline{Z}}^g \\ \text{``} \\ \sum^{3+2g} H\underline{\underline{Z}} \end{matrix} \textcircled{7}$$

Slice SS,  $\sum^7 H_{c_2} \underline{\underline{Z}}$ 

Ex  $\sum^{2\sigma} H_{c_2} \underline{\underline{Z}}$  2-slice?

$\sum^{\sigma-1} H_{c_2} \underline{\underline{Z}}$  0-slice?

$$\textcircled{0} \quad H_{c_2} \underline{\underline{Z}}^\sigma \rightarrow \sum^{\sigma-1} H_{c_2} \underline{\underline{Z}} \rightarrow \sum_{(-2)} H_{c_2} g$$

$$\textcircled{2} \quad \begin{matrix} \sum^{\sigma} H_{c_2} \underline{\underline{Z}} \\ \text{``} \\ \sum^2 H_{c_2} \underline{\underline{Z}} \end{matrix} \rightarrow \sum^{2\sigma} H_{c_2} \underline{\underline{Z}} \rightarrow \sum^{\sigma} H_{c_2} g$$

Slice f. lt. for = Post. fitt. for

$$\sum^{2\sigma} H_{c_2} \underline{\underline{Z}} = \sum^{\sigma} H_{c_2} \underline{\underline{Z}}$$



$$\textcircled{8} \quad \Sigma^4 Hg \rightarrow \Sigma^7 H \underline{\mathbb{Z}}$$

$$\textcircled{7} \quad \Sigma^{3+2g} H \underline{\mathbb{Z}}^\sigma$$

$$\textcircled{2} \quad \Sigma^2 H_{c_2} \underline{\mathbb{Z}} \rightarrow \Sigma^{2\sigma} H_{c_2} \underline{\mathbb{Z}}$$

$$\textcircled{6} \quad H_{c_2} \underline{\mathbb{Z}}^\sigma$$

Examples of slice towers  $\sum^\lambda H_{c_4} \underline{\mathbb{Z}}$

$$\text{Ex } \sum^\lambda H_{c_4} \underline{\mathbb{Z}}$$

- know have 2-slice,  $2=|\lambda|$   
(detected on  $\downarrow_{c_4}^{c_4}$ )

$$\cdot \downarrow_{c_2}^{c_4} \sum^\lambda H_{c_4} \underline{\mathbb{Z}} \simeq \sum^{2\sigma} H_{c_2} \underline{\mathbb{Z}} \text{ has } 0, 2 \text{ slices}$$

$$\cdot P_0^\circ \sum^\lambda H_{c_4} \underline{\mathbb{Z}} = H_{c_4} \underline{\mathbb{Z}} \circ \sum^\lambda H_{c_4} \underline{\mathbb{Z}} = H_{c_4} \underline{B}(2,0)$$

- We found Postnikov filtration

$$P_1 = \sum^2 H_{c_4} \underline{\mathbb{Z}} \rightarrow \sum^\lambda H_{c_4} \underline{\mathbb{Z}} \rightarrow H_{c_4} \underline{B}(2,0) = P_0^\circ$$

Claim  $\sum^2 H_{c_4} \underline{\mathbb{Z}}$  is a 2-slice.

$$\sum^2 H_{C_4} \underline{\mathbb{Z}}$$

$$P_1 = \sum^2 H_{C_4} \underline{\mathbb{Z}} \rightarrow \sum^2 H_{C_4} \underline{\mathbb{Z}}$$

$$\downarrow H_{C_4} B(2,0) = P_0^\circ$$

### Examples of slice towers $\sum^2 H_{C_4} \underline{\mathbb{Z}}$

Claim  $\sum^2 H_{C_4} \underline{\mathbb{Z}}$  is a 2-slice.

- $\sum^2 H_{C_4} \underline{\mathbb{Z}}$  2-connective  $\Rightarrow$  slice 2-connective  $(2 \geq 0)$
- $\sum^2 H_{C_4} \underline{\mathbb{Z}} \leq 2$

$$\downarrow_{C_2}^{C_4} \sum^2 H_{C_4} \underline{\mathbb{Z}} = \sum^2 H_{C_2} \underline{\mathbb{Z}} \text{ a 2-slice.}$$

$$\text{Need } [S^{k\vartheta}, \sum^2 H_{C_4} \underline{\mathbb{Z}}]^{C_4} = 0, \quad 4k > 2.$$

$$[S^8, \sum^2 H_{C_4} \underline{\mathbb{Z}}]^{C_4} = \pi_{-2}(\sum^{-8} H_{C_4} \underline{\mathbb{Z}}) = 0$$

So The Postnikov filtration for  $\sum^2 H_{C_4} \underline{\mathbb{Z}}$  is the slice filtration.

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$$\sum^{\infty} H_{c_0} \mathbb{Z}$$

$$P_i^2 = \sum^2 H_{c_0} \mathbb{Z} \rightarrow \sum^2 H_{c_0} \mathbb{Z}$$

$$\downarrow$$

$$H_{c_0} B(z, 0) = P_0^\circ$$

### The (regular) slice filtration $H = |H|$

Hill-Yarnall  $X \in Sp^G$

Then  $X \geq n \iff \Phi^n X$  is  $\frac{n}{|H|}$ -connective  
 $\forall H \leq G$  vanishes below this

Ex •  $X \geq 1 \iff \downarrow_{e^*}^{C_2} X$  1-connective

$G = C_2$  &  $\Phi^{C_2} X$  1-connective  
 $= 1\text{-connective}$

•  $X \geq 2 \iff \downarrow_{e^*}^{C_2} X$  2-conn,  
&  $\Phi^{C_2} X$  1-conn

## The (regular) slice filtration

$X \geq 2 \Leftrightarrow$   
 $\downarrow_e^{C_2} X$  2-conn,  
 $\nexists \Phi^{C_2} X$  1-conn

$$\underline{\Sigma}^1 H_{C_2} g : \downarrow_e^{C_2} \Sigma^1 H_{C_2} g \simeq * \text{ n-conn. } \forall n$$

$$\overline{\Phi}^{C_2} \Sigma^1 H_{C_2} g \simeq \Sigma^1 \overline{\Phi}^{C_2} H_{C_2} g \simeq \Sigma^1 H \mathbb{F}_2 \text{ 1-conn}$$

$$\Rightarrow \Sigma^1 H_{C_2} g \geq 2.$$

$$\underline{\Sigma}^{-\sigma-\lambda} H_{C_4} \mathbb{Z} : \downarrow_e^{C_4} \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \simeq \Sigma^{-3} H \mathbb{Z} \geq -3$$

$$\overline{\Phi}^{C_2} \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \simeq \Sigma^{-1} \overline{\Phi}^{C_2} H_{C_2} \mathbb{Z} \geq -1$$

$$\overline{\Phi}^{C_4} \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \simeq \overline{\Phi}^{C_4} H_{C_4} \mathbb{Z} \geq 0$$

$$\Rightarrow \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \geq -3$$

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$$\downarrow_e^{c_1} \Sigma^1 H_{C_2} g \cong *$$

$$\Phi^{c_1} \Sigma^1 H_{C_2} g \cong \Sigma^1 H \#_2$$

$$\Rightarrow \Sigma^1 H_{C_2} g \geq 2$$


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$$\downarrow_e^{c_4} \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \cong \Sigma^{-3} H \mathbb{Z} \geq -3$$

$$\Phi^{c_4} \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \cong \Sigma^{-1} \Phi^{c_2} H_{C_2} \mathbb{Z} \geq -1$$

$$\Phi^{c_4} \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \cong \Sigma^{-1} \Phi^{c_4} H_{C_4} \mathbb{Z} \geq 0$$

$$\Rightarrow \Sigma^{-\sigma-\lambda} H_{C_4} \mathbb{Z} \geq -3$$

## RO(G)-grading

$RO(G)$  = real representation ring of  $G$   
 $= \mathbb{Z}\{\text{iso. classes of irreps of } G / \text{IR}\}$  (choices of representatives)

Ex •  $RO(C_2) = \mathbb{Z}\{1, \sigma\}$    •  $RO(C_3) = \mathbb{Z}\{1, \lambda\}$   
•  $RO(C_4) = \mathbb{Z}\{1, \tau, \lambda\}$    •  $RO(C_2 \times C_2) = \mathbb{Z}\{1, P_1, \tau, m^\ast \tau, P_2 \tau\}$

$RO(G)$ -grading  $V, W \in \text{Rep}(G)$   $S^{V-W} = S^V \wedge S^{-W}$

$$\pi_{V-W}(x) = [S^{V-W}, x]^G, \quad \pi_{V-W}^H(x) = [\downarrow_H^G S^{V-W}, \downarrow_H^G x]^H$$

$\Rightarrow RO(G)$ -graded Mackey functor  $\underline{\pi}_* X$

$$\underline{\pi}_{-\sigma} H_{C_2} \mathbb{Z} \cong \underline{\pi}_0 \sum H_{C_2} \mathbb{Z} \cong \underline{g}$$

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$$\text{RO}(G) = \mathbb{Z} \left\{ \begin{array}{l} \text{iso. classes} \\ \text{of irreps of } G \end{array} \right\}$$

$$\text{RO}(C_2) = \mathbb{Z} \{ 1, \tau \}$$

$$\text{RO}(C_4) = \mathbb{Z} \{ 1, \tau, \lambda \}$$

$$\pi_v^H(x) = [\downarrow_H^G S^v, \downarrow_H^G x]^H$$

$$\pi_v X =$$

$$\begin{array}{c} \text{W}(k) \xrightarrow{\pi_v^k(x)} \pi_v^G(x) \\ \text{W}(k) \xrightarrow{\pi_v^H(x)} \pi_v^H(x) \xrightarrow{\sim} \text{W}(H) \\ \text{W}(k) \xrightarrow{\pi_v^e(x)} \pi_v^e(x) \xrightarrow{\sim} G \end{array}$$

## RO(G)-grading (Multiplicative Structure)

$$R \text{ ring in } Sp^G \rightarrow \pi_v(R) \otimes \pi_w(R) \rightarrow \pi_{v \otimes w}(R)$$

$$R \text{ comm ring in } Sp^G \rightsquigarrow \pi_*(R) \text{ graded-comm !}$$

$$\underline{\text{Ex}} \quad G = C_2 \quad \pi_{i+k\tau}(R) \otimes \pi_{j+l\tau}(R) \rightarrow \pi_{i+j+(k+l)\tau}(R)$$

$$\alpha \cdot \beta = (-1)^{ij} \epsilon^{ikl} \beta \cdot \alpha$$

$$\epsilon = \tau w: S^\tau \wedge S^\tau \cong S^\tau \wedge S^\tau$$

$$S^\circ \rightarrow kR \rightarrow H_{C_2} \mathbb{Z} \xrightarrow{\pi_\circ} A \xrightarrow{\epsilon} \mathbb{Z} \cong \mathbb{Z}$$

## RO(G)-grading

R ring in  $S^G$

$$\pi_v(R) \otimes \pi_w(R) \rightarrow \pi_{v \wedge w}(R)$$

$$G = C_2$$

$$\alpha \in \pi_{i+k\tau}(R)$$

$$\beta \in \pi_{j+l\tau}(R)$$

$$\alpha \cdot \beta = (-1)^{ij} \epsilon^{kl} \beta \cdot \alpha$$

$$S^\circ \rightarrow kR \rightarrow H_c \mathbb{Z}$$

$$\epsilon \longmapsto -1$$

Some elements

- $V \in \text{Rep}(G)$ ,  $S^\circ \xrightarrow{\alpha_v} S^V \quad \alpha_v \in \pi_{-v}(S^\circ)$
- For  $\tau \in \text{Rep}(C_2)$   $C_{2+} \rightarrow S^\circ \xrightarrow{\alpha} S^\tau \Rightarrow \begin{cases} \text{im}(T_e^{C_2}) = \ker(\alpha) \\ \ker(R_e^{C_2}) = \text{im}(\alpha) \end{cases}$
- $V \in \text{Rep}(G)$ ,  $u_v \in \pi_{V-v} H_G \mathbb{Z} \cong \tilde{H}(S^{V-H}; \mathbb{Z})$   
 $V = \dim V \quad \text{if } V \text{ orientable}, \quad R_e^G u_V = \pm 1.$
- $G = C_2 \quad 2\tau \text{ orientable} \Rightarrow \exists \quad u_{2\tau} \in \pi_{2-2\tau} H_{C_2} \mathbb{Z}$

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$$S^{\circ} \xrightarrow{a} S^{\sigma}$$

$$\text{im}(T_e^{c_2}) = \ker(a)$$

$$\ker(R_e^{c_2}) = \text{im}(a)$$

$$u_{2\sigma} \in \pi_{z-z\sigma} H_{c_2} \mathbb{Z}$$

$$R u_{z\sigma} = \pm 1 \in \pi_0 H \mathbb{Z}$$

$$g = \begin{matrix} \mathbb{Z}/2 \\ 0 \end{matrix} \quad \underline{\mathbb{Z}} = \begin{matrix} \mathbb{Z} \\ \downarrow \end{matrix} \quad \underline{\mathbb{Z}}^* = \begin{matrix} \mathbb{Z} \\ \downarrow \end{matrix} \\ \underline{\mathbb{Z}}^\sigma = \begin{matrix} 0 \\ \mathbb{Z} \end{matrix} \quad Q\underline{\mathbb{Z}} = \begin{matrix} \mathbb{Z}/2 \\ \mathbb{Z} \end{matrix}$$

	$\uparrow j$	$\mathbb{Z}^*$	$\mathbb{Z}^*$	$\mathbb{Z}^*$
		$Q\mathbb{Z}^*$	$\mathbb{Z}^*$	$\mathbb{Z}^*$
		$\mathbb{Z}^*$	$\mathbb{Z}^*$	$\mathbb{Z}^*$
		$\mathbb{Z}^*$	$\mathbb{Z}^*$	$\mathbb{Z}^*$
		$\mathbb{Z}^*$	$\mathbb{Z}^*$	$\mathbb{Z}^*$
		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
			$\rightarrow n$	
		$\mathbb{Z}^*$	$\mathbb{Z}$	
		$\mathbb{Z}$	$\mathbb{Z}$	
		$\mathbb{Z}^*$	$\mathbb{Z}$	
		$\mathbb{Z}$	$\mathbb{Z}$	
		$\mathbb{Z}^*$	$\mathbb{Z}$	
		$\mathbb{Z}$	$\mathbb{Z}$	

$$\pi_{n-j+j\sigma} H_{c_2} \mathbb{Z}$$

$$\pi_* H_{c_2} \mathbb{Z}$$

$$j \uparrow \quad Q\mathbb{Z} \xrightarrow[\mathbb{Z}]{} \mathbb{Z}^{\frac{\Theta_L}{\alpha^2}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha^4}} \mathbb{Z}^{\frac{\Theta_L}{\alpha^2}}$$

$$Q\mathbb{Z}^{\frac{\Theta_L}{\alpha^2}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha^2}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha^{-1}}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha^{-1}}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha^2}}$$

$$\mathbb{Z}^{\frac{\Theta_L}{\alpha^3}}$$

$$a_2$$

$$a^2_2$$

$$a^3_2$$

$$a^4_2$$

$$\rightarrow n$$

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$$\pi_{n-j+j\sigma} H_{C_2} \underline{\mathbb{Z}}$$

	$Q\mathbb{Z}_{x^2}^{\theta_\infty}$	$\mathbb{Z}_{x^2}^{\theta_\infty}$
	$\mathbb{Z}_{x^2}^{e_\infty}$	$\mathbb{Z}_{x^2}^{e_\infty}$
	$Q\mathbb{Z}_{x^2}^\theta$	$\mathbb{Z}_{x^2}^\theta$
	$\mathbb{Z}_{x^2}^z$	$\mathbb{Z}_{x^2}^z$
	$\mathbb{Z}_{x^2}^r$	$\mathbb{Z}_{x^2}^r$
	$\mathbb{Z}_{x^2}^t$	$\mathbb{Z}_{x^2}^t$
	$\mathbb{Z}_{x^2}^u$	$\mathbb{Z}_{x^2}^u$
	$\mathbb{Z}_{x^2}^v$	$\mathbb{Z}_{x^2}^v$
	$\mathbb{Z}_{x^2}^w$	$\mathbb{Z}_{x^2}^w$
	$\mathbb{Z}_{x^2}^x$	$\mathbb{Z}_{x^2}^x$
	$\mathbb{Z}_{x^2}^y$	$\mathbb{Z}_{x^2}^y$
$a_2$	$\mathbb{Z}_x^r$	$\mathbb{Z}_x^r$
$a_2^2$	$\mathbb{Z}_x^u$	$\mathbb{Z}_x^u$
$a_2^3$	$\mathbb{Z}_x^v$	$\mathbb{Z}_x^v$
$a_2^4$	$\mathbb{Z}_x^w$	$\mathbb{Z}_x^w$
$a_2^5$	$\mathbb{Z}_x^x$	$\mathbb{Z}_x^x$

$$g = \begin{pmatrix} \mathbb{Z}/2 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$$\underline{\mathbb{Z}}^\sigma = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}, Q\underline{\mathbb{Z}} = \begin{pmatrix} \mathbb{Z}/2 \\ \mathbb{Z} \end{pmatrix}$$

RO(G)-grading: The slice spectral sequence

$$\mathbb{E}_2^{s,t} = \pi_{t-s} P_t^t X \Rightarrow \pi_{t-s} X.$$

$$d_r: \pi_n P_t^t X \longrightarrow \pi_{n-1} P_{t+r-1}^{t+r-1} X$$

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$$\pi_{n-j+j\sigma} H_{C_2} \mathbb{Z}$$

	$Q\mathbb{Z}_{x^2}^{\theta_1}$	$\mathbb{Z}_{x^2}^{\theta_2}$
	$\mathbb{Z}_{x^2}^{e_1}$	$\mathbb{Z}_{x^2}^{e_2}$
	$Q\mathbb{Z}_{x^2}^{\theta_3}$	
	$\mathbb{Z}_{x^2}^{e_3}$	
	$\mathbb{Z}_{x^2}^{\sigma}$	
	$\mathbb{Z}_{x^2}^{\tau}$	
	$\mathbb{Z}_{x^2}^{\delta}$	
	$\mathbb{Z}_{x^2}^{\epsilon}$	
$\alpha_2$	$\mathbb{Z}_{x^2}^{\alpha_2}$	
$\alpha_2$		$\mathbb{Z}_{x^2}^{\alpha_2}$
$\alpha_3$	$\mathbb{Z}_{x^2}^{\alpha_3}$	$\mathbb{Z}_{x^2}^{\alpha_3}$

$$g = \begin{pmatrix} \mathbb{Z}/2 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \quad \underline{\mathbb{Z}} = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \quad \underline{\mathbb{Z}}^* = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}$$

$$\underline{\mathbb{Z}}^\sigma = \begin{pmatrix} 0 \\ \mathbb{Z} \end{pmatrix} \quad Q\underline{\mathbb{Z}} = \begin{pmatrix} \mathbb{Z}/2 \\ \mathbb{Z} \end{pmatrix}$$

RO(G)-grading: The slice spectral sequence

$$E_2^{s,V} = \underline{\mathbb{P}}_{V-s} P_V^V X \Rightarrow \underline{\mathbb{P}}_{V-s} X.$$

$$d_r: \underline{\mathbb{P}}_{V-s} P_V^V X \longrightarrow \underline{\mathbb{P}}_{V-s-1} P_{V+r-1}^V X$$

R comm ring in  $Sp^G$

$$E_2^{s,V} \otimes E_2^{s',V'} \longrightarrow E_2^{s+s',V+V'}$$

$$\underline{\mathbb{P}}_{V-s} P_V^V R \otimes \underline{\mathbb{P}}_{V'-s'} P_{V'}^{V'} R \rightarrow \underline{\mathbb{P}}_{V+V'-s-s'} P_{V+V'}^{V+V'} R$$

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$$\pi_{n-j+j\sigma} H_{C_2} \mathbb{Z}$$

$$\begin{array}{c|cc}
& Q\mathbb{Z}_{x^2}^{\theta_1} & \mathbb{Z}_{x^2}^{\theta_2} \\
\hline
\mathbb{Z}_{x^4}^{\theta_1} & \mathbb{Z}_{x^4}^{\theta_2} & \\
Q\mathbb{Z}_{x^2}^{\theta_1} & \mathbb{Z}_{x^2}^{\theta_2} & \\
\mathbb{Z}_{x^2}^{\theta_2} & & \\
\mathbb{Z}_{x^2}^{\theta_1} & & \\
\mathbb{Z}_{x^2}^{\theta_2} & & \\
\hline
\mathbb{Z}_{x^2}^{\theta_1} & & \\
\mathbb{Z}_{x^2}^{\theta_2} & & \\
\hline
\end{array}$$

$$\begin{array}{c|cc}
& \mathbb{Z}_{x^2}^{\alpha_2} & \mathbb{Z}_{x^2}^{\alpha_3} \\
\hline
\mathbb{Z}_{x^2}^{\alpha_2} & \mathbb{Z}_{x^2}^{\alpha_2} & \mathbb{Z}_{x^2}^{\alpha_3} \\
\mathbb{Z}_{x^2}^{\alpha_3} & \mathbb{Z}_{x^2}^{\alpha_3} & \mathbb{Z}_{x^2}^{\alpha_2} \\
\hline
\end{array}$$

$$g = \frac{\mathbb{Z}_{x^2}}{0} \quad \underline{\mathbb{Z}} = \frac{\mathbb{Z}_{x^2}}{\mathbb{Z}} \quad \underline{\mathbb{Z}}^* = \frac{\mathbb{Z}_{x^2}}{\mathbb{Z}_{x^2}}$$

$$\underline{\mathbb{Z}}^* = \frac{0}{\mathbb{Z}_{x^2}} \quad Q\underline{\mathbb{Z}} = \frac{\mathbb{Z}_{x^2}}{\mathbb{Z}_{x^2}}$$

RO(G)-grading: The slice spectral sequence

$$E_2^{s,V} = \pi_{V-s} P_V^\vee X \Rightarrow \pi_{V-s} X.$$

$$d_r: \pi_{V-s} P_V^\vee X \longrightarrow \pi_{V-s-1} P_{V-r-1}^\vee X$$

R comm ring in  $Sp^G$

$$E_2^{s,V} \otimes E_2^{s',V'} \longrightarrow E_2^{s+s',V+V'}$$

$$\pi_{V-s} P_V^\vee R \otimes \pi_{V-s'} P_{V'}^\vee R \rightarrow \pi_{V+V'-s-s'} P_{V+V'}^\vee R$$

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$$\mathbb{H}_{n-j+j\sigma} H_{C_2} \mathbb{Z}$$

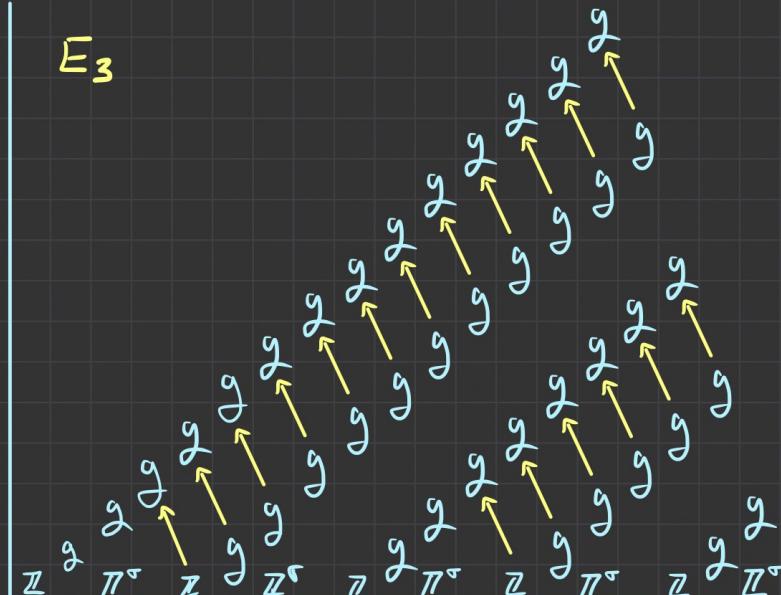
$$\begin{array}{c} Q\mathbb{Z}^{\theta_u}_{x^3} \\ \mathbb{Z}^{\theta_u}_{x^2} \\ Q\mathbb{Z}^{\theta}_{x^3} \\ \mathbb{Z}^{\theta}_{x^2} \\ \mathbb{Z}^{\sigma}_{x^3} \\ \mathbb{Z}^{\sigma}_{x^2} \\ \mathbb{Z}^{\sigma} \\ \mathbb{Z}^{\sigma}_{x^3} \\ \mathbb{Z}^{\sigma}_{x^2} \\ \mathbb{Z}^{\sigma}_{x^1} \\ \mathbb{Z}^{\sigma}_{x^0} \end{array}$$

$$\begin{array}{ccc} g^3 & g^2 & g \\ g^2 & g^1 & g \\ g^1 & g^0 & g^3 \end{array}$$

$$g = \frac{\mathbb{Z}/z}{0} \quad \underline{\mathbb{Z}} = \frac{\mathbb{Z}}{\mathbb{Z}} \quad \underline{\mathbb{Z}}^* = \frac{\mathbb{Z}}{\mathbb{Z}}$$

$$\underline{\mathbb{Z}}^{\sigma} = \frac{\mathbb{Z}}{\mathbb{Z}} \quad Q\underline{\mathbb{Z}} = \frac{\mathbb{Z}/z}{\mathbb{Z}}$$

## $ROCC_2$ -graded SSS for $kIR$

 $E_3$ 

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$$\mathfrak{U}_{n-j+j\sigma} H_{c_2} \mathbb{Z}$$

$$\begin{array}{|c|c|} \hline & Q\mathbb{Z}_{x^3}^{\theta_1} \\ \hline & \mathbb{Z}_{x^4}^{\sigma_1} \\ \hline & Q\mathbb{Z}_{x^3}^{\theta_2} \\ \hline & \mathbb{Z}_{x^4}^{\sigma_2} \\ \hline & \mathbb{Z}_1 \\ \hline \hline & \mathbb{Z}_x \\ \hline & \mathbb{Z}_{x^2} \\ \hline & \mathbb{Z}_{x^3} \\ \hline & \mathbb{Z}_{x^4} \\ \hline & \mathbb{Z}_1 \\ \hline \end{array}$$

$$g = \frac{\mathbb{Z}_{12}}{0} \quad \underline{\mathbb{Z}} = \frac{\mathbb{Z}_{12}}{\mathbb{Z}} \quad \underline{\mathbb{Z}}^* = \frac{\mathbb{Z}_{12}}{\mathbb{Z}}$$

$$\underline{\mathbb{Z}}^* = \frac{\circ}{\mathbb{Z}_{12}} \quad Q\underline{\mathbb{Z}} = \frac{\mathbb{Z}_{12}}{\mathbb{Z}_{12}}$$

$\text{ROCC}_2$ -graded SSS for  $k\mathbb{R}$

$$\mathfrak{U}_* \sum^s H_{c_2} \mathbb{Z} \cong \mathfrak{U}_* H_{c_2} \mathbb{Z} \setminus \{r_i\} \quad r_i \in \mathfrak{U}_S \sum^s H_{c_2} \mathbb{Z}$$

$$E_3^{*,*}$$

$$\begin{array}{|c|c|c|c|} \hline & \mathbb{Z}_1 & \mathbb{Z}_{x^2} & \mathbb{Z}_{x^3} & \mathbb{Z}_{x^4} \\ \hline & g & g & g & g \\ \hline & a^3 g & a^6 g & a^9 g & a^{12} g \\ \hline & a^2 g & a^5 g & a^8 g & a^{11} g \\ \hline & a^1 g & a^4 g & a^7 g & a^{10} g \\ \hline & a^0 g & a^3 g & a^6 g & a^9 g \\ \hline \end{array}$$

$$d_3(u r_i^2) = a^3 r_i^3$$

$$E_3^{*,*+2-2\sigma}$$

$$\begin{array}{|c|c|c|} \hline & \mathbb{Z}_1 & \mathbb{Z}_{x^2} & \mathbb{Z}_{x^3} \\ \hline & g & g & g \\ \hline & a^3 g & a^6 g & a^9 g \\ \hline & a^4 g & a^7 g & a^{10} g \\ \hline & a^5 g & a^8 g & a^{11} g \\ \hline & a^6 g & a^9 g & a^{12} g \\ \hline & a^7 g & a^{10} g & a^{13} g \\ \hline & a^8 g & a^{11} g & a^{14} g \\ \hline & a^9 g & a^{12} g & a^{15} g \\ \hline & a^{10} g & a^{13} g & a^{16} g \\ \hline \end{array}$$

$$d_3(u) = a^3 r_i$$

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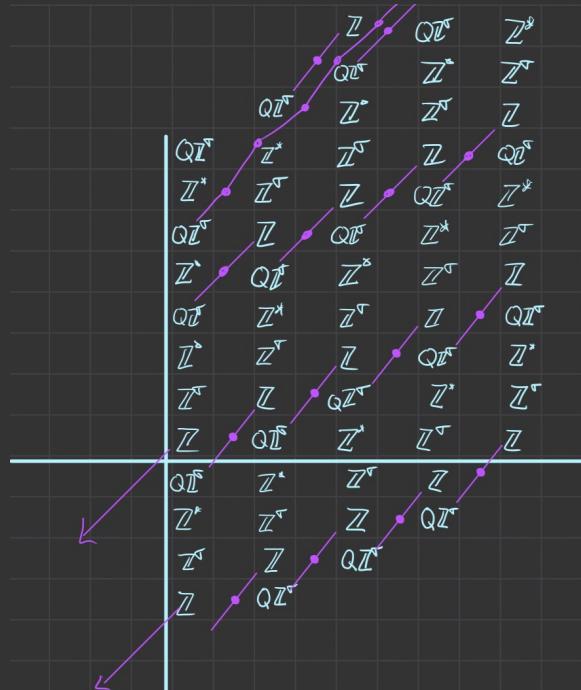
$$\mathbb{H}_{n-j+j\sigma} H_{C_2} \mathbb{Z}$$



$$g = \begin{pmatrix} \mathbb{Z}/2 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \quad \underline{\mathbb{Z}} = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \quad \underline{\mathbb{Z}}^* = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}$$

$$\underline{\mathbb{Z}}^{\sigma} = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \quad Q\underline{\mathbb{Z}} = \begin{pmatrix} \mathbb{Z}/2 \\ \mathbb{Z} \end{pmatrix}$$

## $\text{RO}(C_2)$ -graded SSS for $kR$



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$$\mathbb{H}_{n-j+j\sigma} H_{C_2} \mathbb{Z}$$



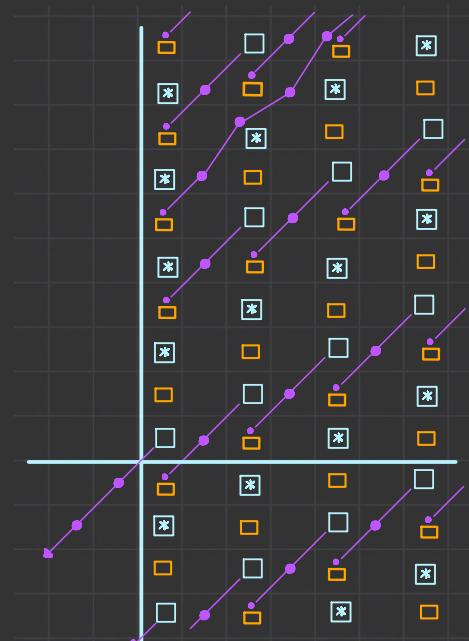
$$g = \frac{\mathbb{Z}/2}{0} \quad \underline{\mathbb{Z}} = \frac{\mathbb{Z}}{\mathbb{Z}} \quad \underline{\mathbb{Z}}^* = \frac{\mathbb{Z}}{\mathbb{Z}}$$

$$\underline{\mathbb{Z}}^\sigma = \frac{\mathbb{Z}}{\mathbb{Z}} \quad Q\underline{\mathbb{Z}} = \frac{\mathbb{Z}/2}{\mathbb{Z}}$$

## $\text{RO}(C_2)$ -graded SSS for $k\mathbb{R}$

Geometric fixed points

$$\underline{\Phi}^{C_2}(k\mathbb{R}) \simeq HF_2[u^2]$$



$$\begin{array}{ccccccccc} \mathbb{Z} & \square & \mathbb{Z}^\sigma & \square & g & \bullet \\ \mathbb{Z}^* & \ast & Q\mathbb{Z}^\sigma & \circ & & & \end{array}$$

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## Lecture 4

- Duality
- Norms
- MUIR, BPIR