The equivariant slice spectral sequence

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Lecture 3

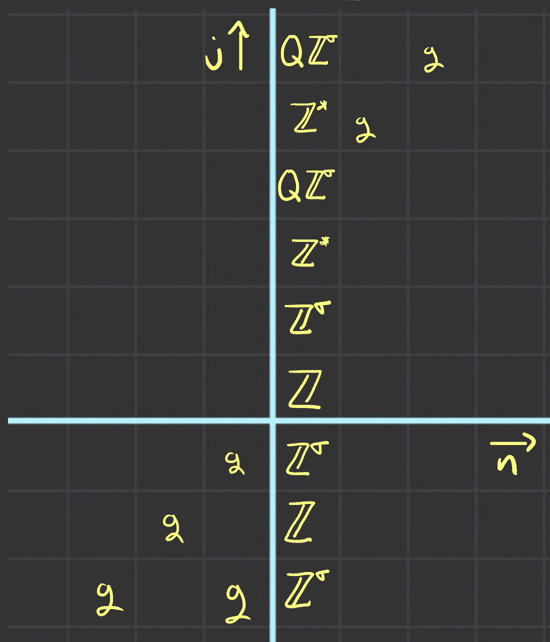
- Examples: $\Sigma^V H_G \underline{\mathbb{Z}}$
  - 8 $\quad \Sigma^{1+3\sigma} H_{\underline{g}} \to \Sigma^7 H\underline{\mathbb{Z}} \to \Sigma^{1+3\sigma} H\underline{\mathbb{Z}}^{\sigma}$ 7
    $\quad\quad \Sigma^4 H_{\underline{g}} = \Sigma^{1+3\sigma} H_{\underline{g}}, \qquad \Sigma^{1+3\sigma} H\underline{\mathbb{Z}}^{\sigma} = \Sigma^{3+2\sigma} H\underline{\mathbb{Z}}$
  - 2 $\ \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\lambda} H_{C_4}\underline{\mathbb{Z}} \to H_{C_4}\underline{B}(2,0)$ 0
- Slice connectivity via geometric fixed points
- The $RO(G)$-grading

Lecture 4

- Duality
- MUIR, BPIR
- Norms

Duality (Anderson, Brown-Comenetz)

$\underline{\pi}_\star H_{C_2}\underline{\mathbb{Z}}$



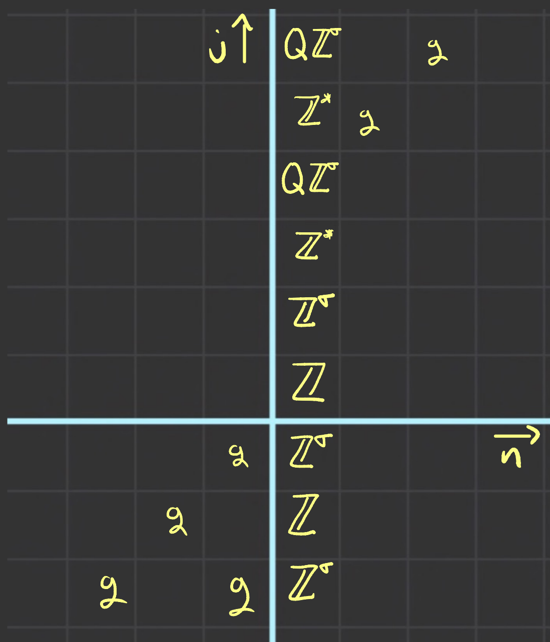
$$\underline{\pi}_{2\sigma-2} H_{C_2}\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^*$$

$$\Sigma^{2-2\sigma} H_{C_2}\underline{\mathbb{Z}} \simeq H_{C_2}\underline{\mathbb{Z}}^*$$

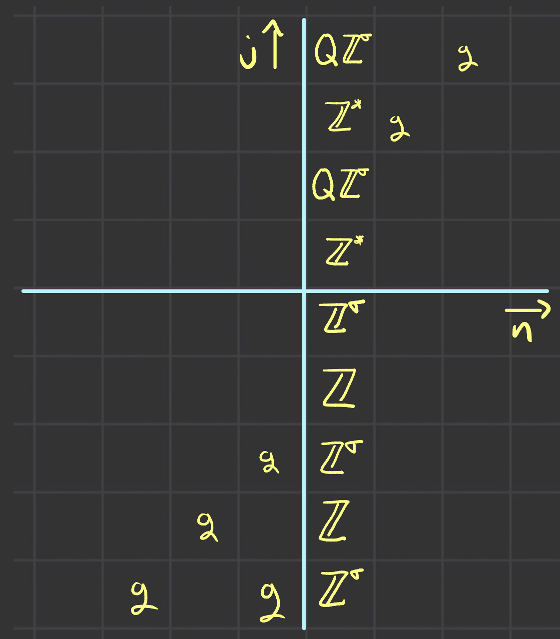
$$\Sigma^{-2\sigma} H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{-2} H_{C_2}\underline{\mathbb{Z}}^*$$

Duality (Anderson, Brown-Comenetz)

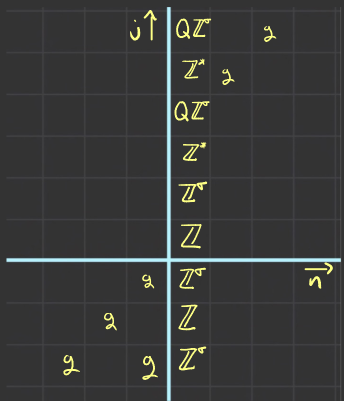
$\underline{\pi}_\star H_{C_2}\underline{\mathbb{Z}}$



$\underline{\pi}_\star H_{C_2}\underline{\mathbb{Z}}^* \cong \underline{\pi}_{\star+2\sigma-2} H_{C_2}\underline{\mathbb{Z}}$



$\underline{\pi}_\star H_{C_2}\underline{\mathbb{Z}}$



Duality (Anderson, Brown-Comenetz)

$\exists\ I_{\mathbb{Q}/\mathbb{Z}} \in Sp$ and $H\mathbb{Q} = I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$ w/

$$\pi_n F(X, I_{\mathbb{Q}/\mathbb{Z}}) = \mathrm{Hom}(\pi_{-n} X, \mathbb{Q}/\mathbb{Z})$$

$$\pi_n F(X, I_{\mathbb{Q}}) = \mathrm{Hom}(\pi_{-n} X, \mathbb{Q})$$

Define $I_{\mathbb{Z}} = \mathrm{fib}(I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}})$.

$$\underset{I_{\mathbb{Z}}X}{F(X, I_{\mathbb{Z}})} \to \underset{I_{\mathbb{Q}}X}{F(X, I_{\mathbb{Q}})} \to \underset{I_{\mathbb{Q}/\mathbb{Z}}X}{F(X, I_{\mathbb{Q}/\mathbb{Z}})}$$

Anderson dual

Brown-Comenetz dual

Duality (Anderson, Brown-Comenetz)

Anderson

$I_{\mathbb{Z}} X = F(X, I_{\mathbb{Z}})$

$\downarrow$

$I_{\mathbb{Q}} X = F(X, I_{\mathbb{Q}})$

$\downarrow$

$I_{\mathbb{Q}/\mathbb{Z}} X = F(X, I_{\mathbb{Q}/\mathbb{Z}})$

Brown-Comenetz

$M \in Ab$

Prop 1) M torsion $\Rightarrow$ $I_{\mathbb{Q}/\mathbb{Z}} HM \simeq HM$

2) M torsion $\Rightarrow$ $I_{\mathbb{Z}} HM \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} HM = \Sigma^{-1} HM$

3) M torsion-free $\Rightarrow$ $I_{\mathbb{Z}} HM \simeq HM$

$$M_{tor} \hookrightarrow M \twoheadrightarrow M_{free}$$

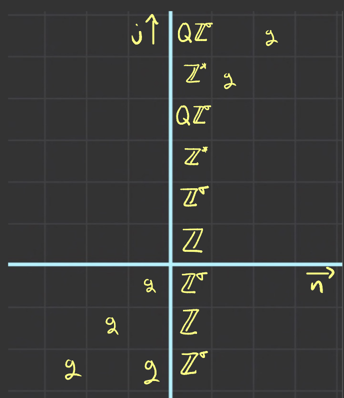
$$\begin{array}{ccccc} I_{\mathbb{Z}} HM_{free} & \to & I_{\mathbb{Z}} HM & \to & I_{\mathbb{Z}} HM_{tor} \\ \| & & & & \| \\ HM_{free} & & & & \Sigma^{-1} HM_{tor} \end{array}$$

$I_{\mathbb{Q}/\mathbb{Z}} HM_{tor} \simeq HM_{tor}$

$I_{\mathbb{Z}} HM_{tor} \simeq \Sigma^{-1} HM_{tor}$

$I_{\mathbb{Z}} HM_{free} \simeq HM_{free}$

$\pi_\star H_{C_2}\underline{\mathbb{Z}}$



Duality (Anderson, Brown-Comenetz)

Works equivariantly too.

- $I_{\mathbb{Z}} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^*$
- $I_{\mathbb{Z}} H\underline{\mathbb{Z}}^\sigma \simeq H\underline{\mathbb{Z}}^\sigma$
- $I_{\mathbb{Q}/\mathbb{Z}} Hg \simeq Hg$

[Ricka]

$$Hg \to H\mathbb{Q}\mathbb{Z}^\sigma \to H\underline{\mathbb{Z}}^\sigma$$

$$\rightsquigarrow$$

$$I_{\mathbb{Z}} Hg \leftarrow I_{\mathbb{Z}} H\mathbb{Q}\mathbb{Z}^\sigma \leftarrow I_{\mathbb{Z}} H\underline{\mathbb{Z}}^\sigma$$

$$I_{\mathbb{Z}} Hg = \Sigma^{-1} Hg \qquad I_{\mathbb{Z}} H\underline{\mathbb{Z}}^\sigma = H\underline{\mathbb{Z}}^\sigma$$

Exercise $I_{\mathbb{Z}} H\mathbb{Q}\mathbb{Z}^\sigma \simeq \Sigma^{\sigma-1} H\underline{\mathbb{Z}}$

Duality (Anderson, Brown-Comenetz)

Duality & the slice filtration

Brown-Comenetz:

Prop (Ullman) $P^n_k(I_{\mathbb{Q}/\mathbb{Z}} X) \simeq I_{\mathbb{Q}/\mathbb{Z}} P^{-k}_{-n} X$

$I_{\mathbb{Z}} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^*$

$I_{\mathbb{Z}} H\underline{\mathbb{Z}}^{\sigma} \simeq H\underline{\mathbb{Z}}^{\sigma}$

$I_{\mathbb{Z}} Hg \simeq \Sigma^{-1} Hg$

Using Anderson Duality: $\Sigma^4 Hg = \Sigma^{1+3\sigma} Hg \to \Sigma^7 H\underline{\mathbb{Z}} \to \Sigma^{1+3\sigma} H\underline{\mathbb{Z}}^{\sigma}$

$\rightsquigarrow I_{\mathbb{Z}} \Sigma^{1+3\sigma} H\underline{\mathbb{Z}}^{\sigma} \to I_{\mathbb{Z}} \Sigma^7 H\underline{\mathbb{Z}} \to I_{\mathbb{Z}} \Sigma^4 Hg$

$-7 \quad \Sigma^{-1-3\sigma} H\underline{\mathbb{Z}}^{\sigma} \qquad \Sigma^{-7} H\underline{\mathbb{Z}}^* \qquad \Sigma^{-4-1} I_{\mathbb{Q}/\mathbb{Z}} Hg$

$\Sigma^{-5-2\sigma} H\underline{\mathbb{Z}} \qquad \Sigma^{-5} Hg \quad -10$

$\Sigma^{2\sigma}$

$-3 \quad \Sigma^{-1-\sigma} H\underline{\mathbb{Z}}^{\sigma} \to \Sigma^{-3} H\underline{\mathbb{Z}} \to \Sigma^{-3} Hg \quad -6$

4/

$$P^n_k(I_{\mathbb{Q}/\mathbb{Z}}X) \simeq I_{\mathbb{Q}/\mathbb{Z}} P^{-k}_{-n} X$$

8 7

$$\Sigma^4 H\underline{g} \to \Sigma^7 H\underline{\mathbb{Z}} \to \Sigma^{1+3\rho} H\underline{\mathbb{Z}}^\sigma$$

$$\rightsquigarrow I_{\mathbb{Z}}$$

-7 -10

$$\Sigma^{-1-3\rho} H\underline{\mathbb{Z}}^\sigma \to \Sigma^{-5-2\sigma} H\underline{\mathbb{Z}} \to \Sigma^{-5} H\underline{g}$$

$$\rightsquigarrow \Sigma^{2\rho}$$

-3 -6

$$\Sigma^{-1-\rho} H\underline{\mathbb{Z}}^\sigma \to \Sigma^{-3} H\underline{\mathbb{Z}} \to \Sigma^{-3} H\underline{g}$$

$M U\mathbb{R}$, $BP\mathbb{R}$

$MU\mathbb{R} \in Sp^{C_2}$ Landweber '60s

$BP\mathbb{R} \in Sp^{C_2}$ Araki '70s

$\pi_\star BP\mathbb{R}$ Araki, Hu-Kriz '90s

Prop [Hu-Kriz] $\Phi^{C_2} BP\mathbb{R} \simeq H\mathbb{F}_2$

$\underline{\pi}_{n\rho} BP\mathbb{R}$ is <u>constant</u>

$$\pi_* BP \cong \mathbb{Z}_{(2)}[v_1, v_2, \ldots] \rightsquigarrow \pi_{*\rho} BP\mathbb{R} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \ldots]$$

$$|v_i| = (2^i - 1)2 \qquad |\bar{v}_i| = (2^i - 1)\rho$$

Note Since $\Phi BP\mathbb{R} \simeq H\mathbb{F}_2$, all $\bar{v}_i$'s a-torsion

MUR, BPR

$\Phi^{C_2} BP\mathbb{R} \simeq H\mathbb{F}_2$

$\underline{\pi}_{n\rho} BP\mathbb{R} \cong \underline{\mathbb{Z}}_{(2)}^{N}$

$\pi_{*\rho} BP\mathbb{R} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \dots]$

$|\bar{v}_i| = (2^{i-1})\rho$

Slice Theorem [HHR] The nontrivial slices of $BP\mathbb{R}$:

$$P^{2n}_{2n} BP\mathbb{R} \simeq \bigvee_{\substack{\text{monomials} \\ \text{in } (v_1, v_2, \dots)}} \Sigma^{n\rho} H\underline{\mathbb{Z}}_{(2)}$$

Ex

$$P^2_2 BP\mathbb{R} \simeq \Sigma^{\rho} H\underline{\mathbb{Z}}_{(2)}\{\bar{v}_1\}$$

$$P^4_4 BP\mathbb{R} \simeq \Sigma^{2\rho} H\underline{\mathbb{Z}}_{(2)}\{\bar{v}_1^2\}$$

$$P^6_6 BP\mathbb{R} \simeq \Sigma^{3\rho} H\underline{\mathbb{Z}}_{(2)}\{\bar{v}_1^3, \bar{v}_2\}$$

$RO(C_2)$-graded SSS for $k\mathbb{R}$

$\Phi^{C_2} BP\mathbb{R} \simeq H\mathbb{F}_2$

$\underline{\pi}_{n\rho} BP\mathbb{R} \cong \underline{\mathbb{Z}}_{(2)}^{N}$

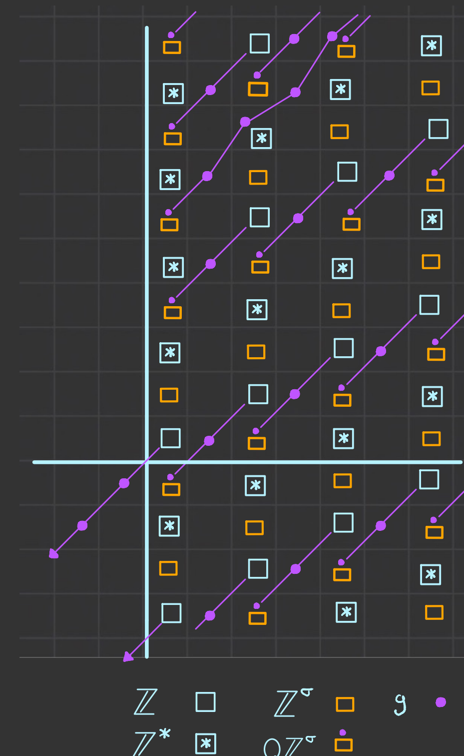
$\pi_{*\rho} BP\mathbb{R} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \dots]$

$|\bar{v}_i| = (2^{i}-1)\rho$

$P^{2n}_{2n} BP\mathbb{R} \simeq \bigvee_{\text{monomials}} \Sigma^{n\rho} H\underline{\mathbb{Z}}_{(2)}$

Geometric fixed points

$$\Phi^{C_2}(k\mathbb{R}) \simeq H\mathbb{F}_2[u^2]$$



$RO(C_2)$-graded SSS for $BP\mathbb{R}$

$\Phi^{C_2} BP\mathbb{R} \simeq H\mathbb{F}_2$

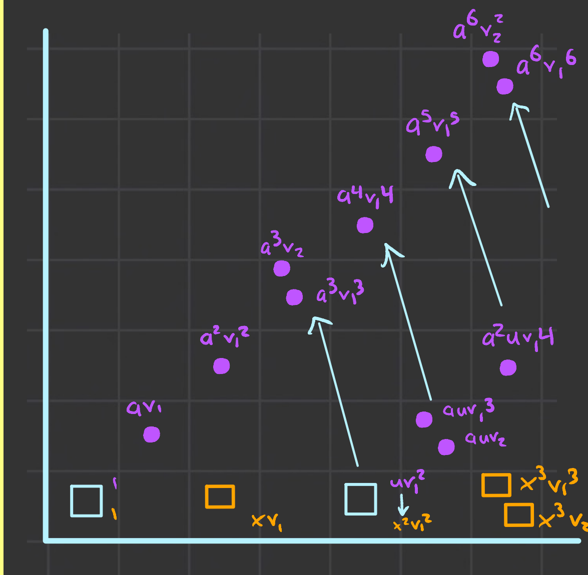
$\underline{\pi}_{n\rho} BP\mathbb{R} \cong \underline{\mathbb{Z}}^N_{(2)}$

$\pi_{*\rho} BP\mathbb{R} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \ldots]$

$|\bar{v}_i| = (2^{i-1})\rho$

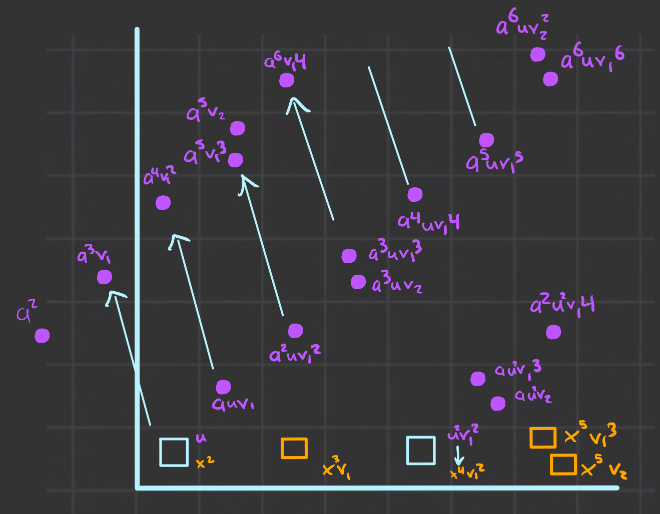
$P^{2n}_{2n} BP\mathbb{R} \simeq \bigvee_{\text{monomials}} \Sigma^{n\rho} H\underline{\mathbb{Z}}_{(2)}$

$E_3^{*,*}$



$d_3(uv_1^2) = a^3 v_1^3$

$E_3^{*,*+2-2\sigma}$



$d_3(u) = a^3 v_1$

$RO(C_2)$-graded SSS for $BP\mathbb{R}$

$\Phi^{C_2} BP\mathbb{R} \simeq H\mathbb{F}_2$

$\underline{\pi}_{n\rho} BP\mathbb{R} \cong \underline{\mathbb{Z}}_{(2)}^{N}$

$\pi_{*\rho} BP\mathbb{R} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \dots]$

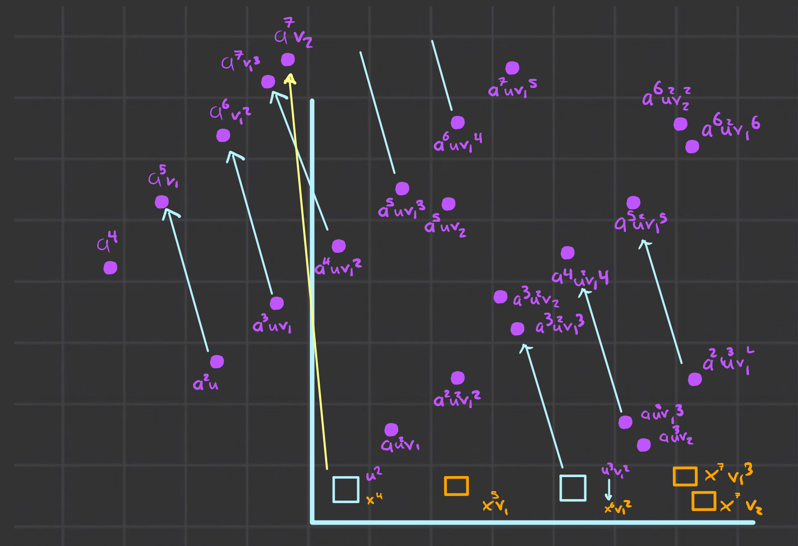
$|\bar{v}_i| = (2^{i-1})\rho$

$P^{2n}_{2n} BP\mathbb{R} \simeq \bigvee_{\text{monomials}} \Sigma^{n\rho} H\underline{\mathbb{Z}}_{(2)}$

$d_3(u) = a^3 v_1$

$E_7^{*,*+4-4\sigma}$

$d_7(u^2) = a^7 v_2$



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$\Phi^{C_2} BP\mathbb{R} \simeq H\mathbb{F}_2$

$\underline{\pi}_{n\rho} BP\mathbb{R} \cong \underline{\mathbb{Z}}_{(2)}^{N}$

$\pi_{*\rho} BP\mathbb{R} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \ldots]$

$|\bar{v}_i| = (2^{i-1})\rho$

$P^{2n}_{2n} BP\mathbb{R} \simeq \bigvee_{\text{monomials}} \Sigma^{n\rho} H\underline{\mathbb{Z}}_{(2)}$

$RO(C_2)$-graded SSS for $BP\mathbb{R}$

Slice Differential Theorem [HHR]

- $d_3(u) = a^3\bar{v}_1$
- $d_7(u^2) = a^7\bar{v}_2$
- $d_{15}(u^4) = a^{15}\bar{v}_3$
- $d_{31}(u^8) = a^{31}\bar{v}_4$

$$\boxed{d_{2^{n+2}-1}(u^{2^n}) = a^{2^{n+2}-1}\bar{v}_{n+1}}$$

What survives?

- $a^k \quad \forall k \geq 0$
- $2u^k \quad \forall k \geq 0$
- $\bar{v}_n \quad n \geq 1$ (truncated a-tower)
- $u^2\bar{v}_1 \qquad u^4\bar{v}_2 \qquad \ldots$

$\cancel{d_7}$: $a^7\bar{v}_1\bar{v}_2 = 0$ $\qquad$ $\cancel{d_{15}}$: $a^{15}\bar{v}_2\bar{v}_3 = 0$

Slice Differential Theorem

$$\boxed{d_{2^{n+2}-1}(u^{2^n}) = a^{2^{n+2}-1}\bar{v}_{n+1}}$$

Norms $Sp^H \to Sp^G$

Induction (additive)

$$Top_{H,*} \xrightarrow{\uparrow_H^G} Top_{G,*}$$

$$\uparrow_H^G X = G_+ \wedge_H X = \bigvee_{G/H} X$$

Norm (multiplicative induction)

$$Top_{H,*} \xrightarrow{N_H^G} Top_{G,*}$$

$$N_H^G X = \bigwedge_{G/H} X$$

Induction

$\uparrow_H^G x = \bigvee_{G/H} x$

Norm

$N_H^G x = \bigwedge_{G/H} x$

Norms $Sp^H \longrightarrow Sp^G$

If $C_2 \leq G$, have $Sp^{C_2} \xrightarrow{N_{C_2}^G} Sp^G$

$\boxed{G = C_{2^n}}$

$MU\mathbb{R}^{((G))} := N_{C_2}^G MU\mathbb{R}$

$BP\mathbb{R}^{((G))} := N_{C_2}^G BP\mathbb{R}$

Slice theorem $\rightsquigarrow P_{2k}^{2k} MU\mathbb{R}^{((G))} \simeq \uparrow_H^G S^{k\rho} \wedge H_G\underline{\mathbb{Z}}$, $H \neq e$

$MU\mathbb{R}^{((C_8))}[D^{-1}]$ $\qquad D \in \pi_{19\rho} MU\mathbb{R}^{((C_8))}$

Show higher Kervaire classes vanish here and would be detected if nonzero

Thanks!