eCHT Minicourse
The Slice Spectral Sequence
Problem Set 2
Spring 2022

1. Show that $\Sigma^{\sigma} H_{C_2}\underline{g} \simeq H_{C_2}\underline{g}$, and use this to compute $\Phi^{C_2} H_{C_2}\underline{g}$.

2. Show that $\Phi^{C_4} X \simeq \Phi^{C_4/C_2} X^{C_2}$. (Hint: show that, if $p\colon C_4 \longrightarrow C_4/C_2$ denotes the quotient map, then $p^*\widetilde{E\mathcal{P}_{C_2}}$ is a model for $\widetilde{E\mathcal{P}_{C_4}}$.)

3. For $G = C_2$, show that a map of C_2-spectra $f\colon X \longrightarrow Y$ is a $\underline{\pi}_*$-isomorphism if and only if it yields an isomorphism of $RO(C_2)$-graded homotopy groups. (Hint: Use the cofber sequence $C_{2+} \to S^0 \to S^{\sigma}$.)

4. (C_4-slice towers)

   (a) Use Figure 3 of HHR's article on the C_4-analogue of real K-theory to show that $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is an n-slice for $0 \le n \le 4$.

   (b) The C_4-Mackey functor $\underline{\mathbb{Z}}(2,1)$ is displayed to the right. Show that there is a short exact sequence of C_4-Mackey functors

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$$

$$\underline{\mathbb{Z}}(2,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g} = \phi^*_{C_4}\mathbb{Z}/2.$$

   (c) Show that

$$P^8_8 = \Sigma^2 H_{C_4}\underline{g} \longrightarrow \Sigma^5 H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{3+2\sigma} H_{C_4}\underline{\mathbb{Z}} = P^5_5$$

   is the slice tower for $\Sigma^5 H_{C_4}\underline{\mathbb{Z}}$. (Hint: Figure 6 of the C_4-article of HHR tells you that $\Sigma^{2-2\sigma} H_{C_4}\underline{\mathbb{Z}} \simeq H_{C_4}\underline{\mathbb{Z}}(2,1)$.)

5. We described the Mackey functors $\underline{\pi}_n k\mathbb{R}$, for $n \ge 0$. This then yields the Mackey functors $\underline{\pi}_n K\mathbb{R}$, for $n \in \mathbb{Z}$, using the 8-fold periodicity of $K\mathbb{R}$. Use the ρ-periodicity of $K\mathbb{R}$ to write down the $RO(C_2)$-graded Mackey functors $\underline{\pi}_\star K\mathbb{R}$. Display these in a chart, and display the nontrivial a-multiplications.

6. Use the equivalences

$$HQ\underline{\mathbb{Z}}^{\sigma} \simeq \Sigma^{3-3\sigma} H\underline{\mathbb{Z}} \qquad \text{and} \qquad H\underline{\mathbb{Z}}^* \simeq \Sigma^{2-2\sigma} H\underline{\mathbb{Z}}$$

   to show that

$$I_{\mathbb{Z}} HQ\underline{\mathbb{Z}}^{\sigma} \simeq \Sigma^{\sigma-1} H\underline{\mathbb{Z}}.$$