Lecture 29

Nov. 2 2011

(1)

This Friday's Lecture: in 100 Noyes

Last time: Change of coordinates for triple integrals.

Given transformation $T: (s,t,u)\text{-plane} \longrightarrow (x,y,z)\text{-plane}$

define the Jacobian of T by $J(T) = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}$

Then $\iiint_R f(x,y,z)\,dV$

$$= \iiint_S f(x(s,t,u), y(s,t,u), z(s,t,u))\,|J(T)|\,ds\,dt\,du$$

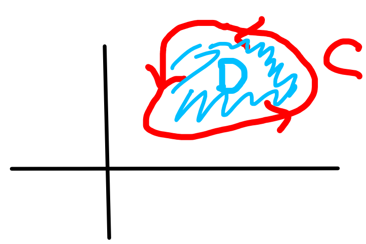
Where T transforms S to R. $(R = T(S))$

Today: Green's theorem. (§16.4)

$\rightsquigarrow$ Fundamental theorem for double integrals

$\vec{F}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ vector field, $\vec{F}(x,y) = (P(x,y), Q(x,y))$

D region in $\mathbb{R}^2$ with boundary curve C (oriented counterclockwise)



Green's Theorem

also written $P\,dx + Q\,dy$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

Sometimes written ∂D

Why is this like Fundamental Theorem?

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

boundary of $[a,b]$

Says 1-dim integral of derivative = 0-dim integral of function

Note: If $F = \nabla f$, then

$$\int_C F \cdot d\vec{r} = \iint_D \frac{\partial}{\partial x}\frac{\partial}{\partial y} f - \frac{\partial}{\partial y}\frac{\partial}{\partial x} f \, dA = 0.$$

$$\| $$

$$f(\vec{r}(b)) - f(\vec{r}(a)) = 0 \quad (\vec{r}(a) = \vec{r}(b))$$

Both sides are 0.

Why true more generally? Take D = disk of radius d,

$\partial D = C$ = circle of radius d,

Want $$\int_C P\,dx + Q\,dy \overset{?}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

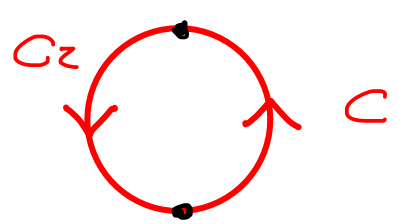
Show $$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\, dA.$$

RHS: have $\frac{\partial Q}{\partial x}$ so want to integrate wrt x.

$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_{-d}^{d}\int_{-\sqrt{d^2-y^2}}^{\sqrt{d^2-y^2}} \frac{\partial Q}{\partial x}\, dx\, dy$$

Fund. thm of Calc $\rightarrow$ $$= \int_{-d}^{d} Q(\sqrt{d^2-y^2}, y) - Q(-\sqrt{d^2-y^2}, y)\, dy$$

LHS: Parametrize C wrt y

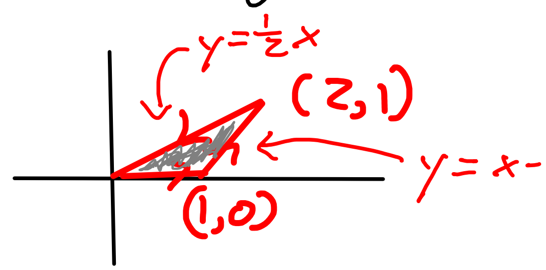


$$\int_C Q\,dy = \int_{C_1} Q\,dy + \int_{C_2} Q\,dy$$

$$= \int_{-d}^{d} Q(\sqrt{d^2-y^2}, y)\,dy + \int_{d}^{-d} Q(-\sqrt{d^2-y^2}, y)\,dy$$

They match up! Similarly for pieces involving P.

Ex $F(x,y) = (x^2, xy)$



Use Green's theorem to find

$\int_C F\cdot d\vec{r}$. Green $\Rightarrow \int_C F\cdot d\vec{r} = \iint_D y - 0\,dA$

$$= \int_0^1 \int_{2y}^{y+1} y\,dx\,dy = \int_0^1 y(y+1-2y)\,dy = \boxed{\tfrac{1}{6}}$$

We replaced complicated line integral (sum of 3 integrals) by easier double integral.

Another use: computing area by line integrals.

Know area$(D) = \iint_D 1\,dA$.

So $\underline{\text{if}}$ $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then

$$\text{area}(D) = \int_{\partial D} F\cdot d\vec{r}$$

Examples 1) $Q(x,y) = x$ $\quad P(x,y) =$ function of x

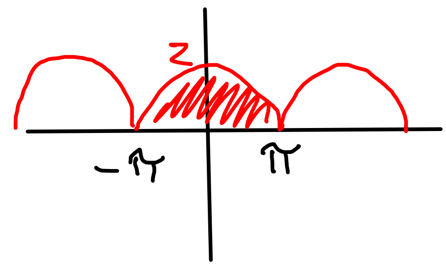
2) $Q(x,y) =$ function of y $\quad P(x,y) = -y$

3) Mixture: $Q(x,y) = kx \quad P(x,y) = (k-1)y$

Then area can be computed as

$$\text{area}(D) = \int_{\partial D} (k-1)y\,dx + kx\,dy$$

Example Find area under the cycloid param. by $(t+\sin t,\ 1+\cos t)$ between $-\pi \le t \le \pi$.



Write $\partial D = \underbrace{C_1}_{x\text{-axis}} \cup \underbrace{C_2}_{\text{cycloid}}$

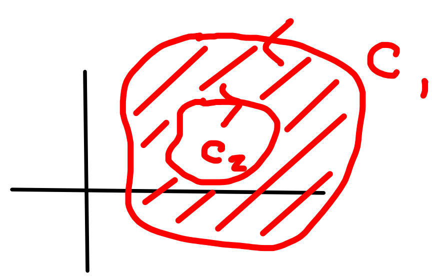
Then $F(x,y) = (-y, 0)$ will be 0 on C_1.

$$\text{area}(D) = \cancel{\int_{C_1} (-y,0)\cdot d\vec{r}} + \int_{C_2} (-y,0)\cdot d\vec{r}$$

$$= -\int_{\pi}^{-\pi} (1+\cos t, 0)\cdot(1+\cos t, -\sin t)\,dt$$

$$= -\int_{\pi}^{-\pi} (1+\cos t)^2\,dt = \dots = 3\pi$$

Green's theorem also applies to non-simply conn. domains. Each boundary piece should be parametrized with the domain on the left. So inner curves should be parametrized clockwise



$$\int_{C_1} F\cdot d\vec{r} + \int_{C_2} F\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$$

Application: Recall the vector field

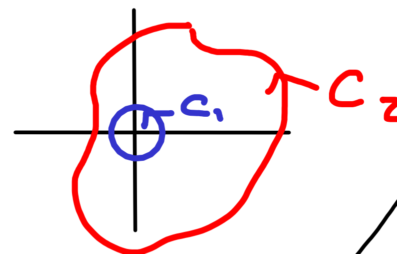
$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. This satisfies $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

but is not conservative since $\int_C F\cdot d\vec{r} = 2\pi \neq 0$

unit circle

Now let C_2 be any curve around origin, oriented counterclockwise. Fit a small circle C_1 inside C_2.

Let D be the region in between.



Then $\int_{C_2} F\cdot d\vec{r} - \int_{C_1} F\cdot d\vec{r}$

$$= \int_{C_2} F\cdot d\vec{r} + \int_{-C_1} F\cdot d\vec{r} \overset{\text{Green}}{=} \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0}\, dA$$

opposite orientation

So $\int_{C_2} F\cdot d\vec{r} = \int_{C_1} F\cdot d\vec{r} \overset{\text{previous calculation}}{=} 2\pi$

for any loop C_2. (C_2 must wind around $(0,0)$ only once).

Actually $\frac{1}{2\pi}\int_{C_2} F\cdot d\vec{r} =$ the "winding number" of the loop C_2 around the origin.