1. Suppose $f(x, y)$ has values and partial derivatives as in the table at right. Find all the critical points you can from the given data and classify them into local mins, local maxes, and saddles. **(3 points)**

(x,y)	f	f_x	f_y	f_{xx}	f_{xy}	f_{yy}
(0,0)	0	0	0	−2	1	−3
(1,0)	2	−1	1	1	0	2
(2,1)	5	0	0	1	3	2

Crit pts where $\nabla f = 0$

(0,0):

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 6-1 = 5 > 0$$

Since $f_{xx} = -2 < 0$, this is a local max.

(2,1)

$$\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = 2-9 = -7 < 0$$

So its a saddle

Local mins (if any):

Local maxes (if any): (0,0)

Saddles (if any): (2,1)

2. Find an equation of the tangent plane to the surface defined by $\underbrace{3x^2 + xy + 2yz}_{f(x,y,z)} = 8$ at the point $(1, 1, 2)$. **(3 points)**

Surface is $f = 8$ and

$$\nabla f = (6x+y, x+2z, 2y)$$

so the normal to the tangent plane is

$$\vec{n} = \nabla f(1,1,2) = (7,5,2)$$

Thus the eqn for the plane is

$$0 = \vec{n} \cdot ((x,y,z)-(1,1,2)) = 7(x-1)+5(y-1)+2(z-2)$$
$$= 7x+5y+2z-16$$

Equation: $\boxed{7}\,x + \boxed{5}\,y + \boxed{2}\,z = \boxed{16}$

3. Let $f(x, y) = xy + 1$. Let C be the curve defined by $x^2 + 4y^2 = 8$.

(a) Find the maximum value M and minimum value m achieved by $f(x, y)$ on the curve C. **(5 points)**

Set $g = x^2 + 4y^2$ so that $C = \{g = 8\}$. Then

$$\nabla f = (y, x) = \lambda \nabla g = \lambda (2x, 8y)$$

and so we want to solve

$$y = \lambda 2x \quad \text{and} \quad x = \lambda 8y \quad \text{and} \quad x^2 + 4y^2 = 8.$$

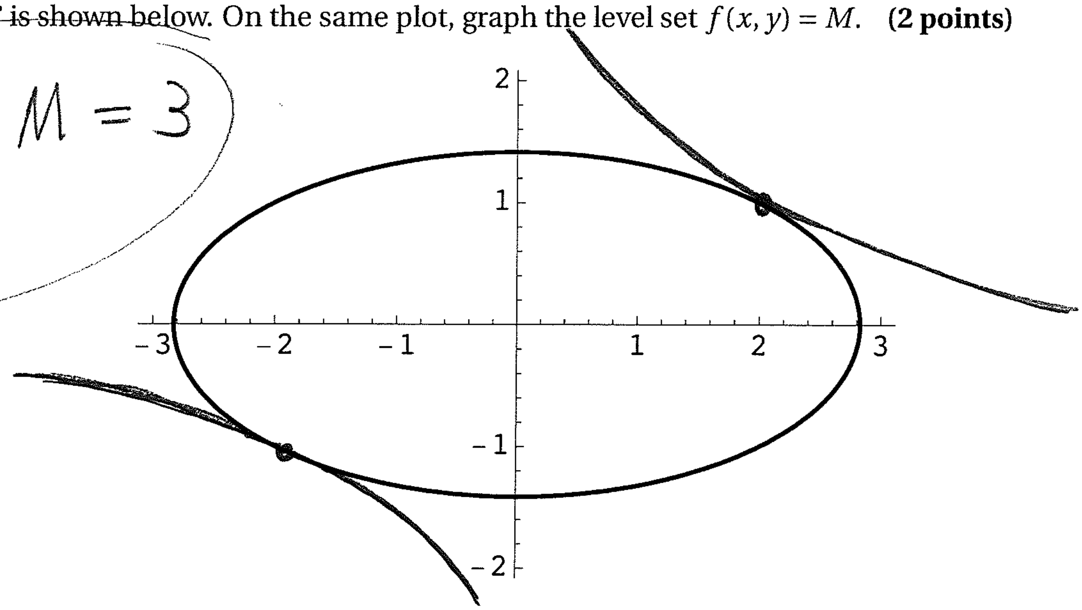
These have no solutions with either x or $y = 0$, so we can divide and find

$$\frac{y}{2x} = \lambda = \frac{x}{8y} \Rightarrow 8y^2 = 2x^2 \Rightarrow 4y^2 = x^2$$

Combining with $x^2 + 4y^2 = 8$, this gives $4y^2 + 4y^2 = 8$ $\Rightarrow y = \pm 1 \Rightarrow x = \pm 2$. Since $f(2,1) = f(-2,-1) = 3$ and $f(2,-1) = f(-2,1) = 1$ we get: $M =$ 3 $m =$ −1

(b) The curve C is shown below. On the same plot, graph the level set $f(x, y) = M$. **(2 points)**

$f(x,y) = M = 3$
$\|$
$xy + 1$
$\Downarrow$
$xy = 2$



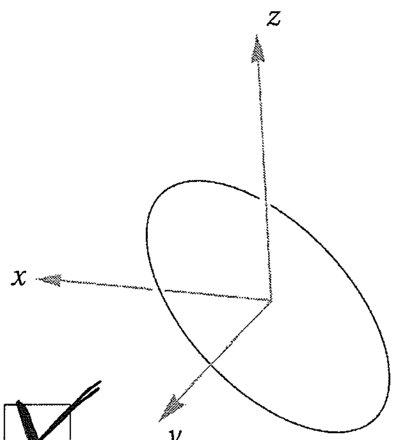
(c) What two properties does the curve C have that guarantee $f(x, y)$ has absolute maximum and minimum values on C? **(1 point each)**

Closed | bounded

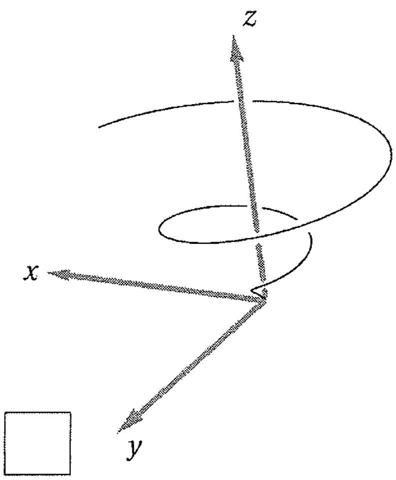
4. Consider the space curve C parameterized by

$$\mathbf{r}(t) = \left(\underset{x}{\cos t}, \underset{y}{\sqrt{2}\sin t}, \underset{z}{\cos t}\right) \quad \text{for } 0 \le t \le 2\pi.$$

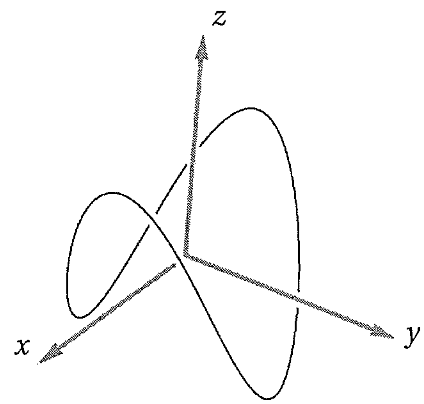
(a) Mark the correct sketch of C below: **(2 points)**



☑



☐



☐

(b) Evaluate the line integral $\int_C (yz+1)\,ds$. **(5 points)**

$$\vec{r}'(t) = (-\sin t, \sqrt{2}\cos t, -\sin t)$$

$$|\vec{r}'(t)|^2 = \sin^2 t + 2\cos^2 t + \sin^2 t = 2$$

$$|\vec{r}'(t)| = \sqrt{2}$$

$$\int_C (yz+1)\,ds = \int_0^{2\pi} (\sqrt{2}\sin t\cos t + 1)\,|\vec{r}'(t)|\,dt$$

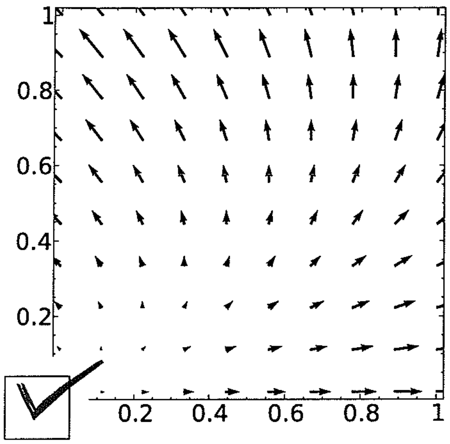
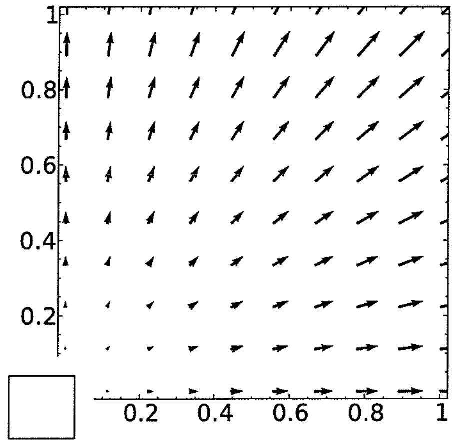
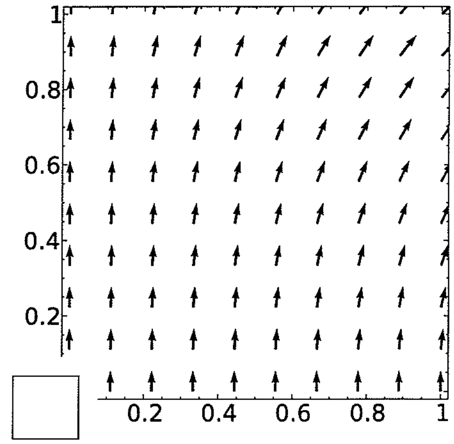
$$= \int_0^{2\pi} 2\sin t\cos t + \sqrt{2}\,dt = \sin^2 t + \sqrt{2}t \Big|_{t=0}^{2\pi}$$

$$= 0 + \sqrt{2}(2\pi) - (0+0) = 2\sqrt{2}\pi$$

$$\boxed{\int_C (yz+1)\,ds = 2\sqrt{2}\pi}$$

5. Consider the vector field $\mathbf{F} = \langle x - y, y \rangle$.

(a) Mark the picture of $\mathbf{F}$ below: **(2 points)**

(b) Consider the curve C parameterized by $\mathbf{r}(t) = (t^2, t)$ for $0 \le t \le 1$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(4 points)**

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

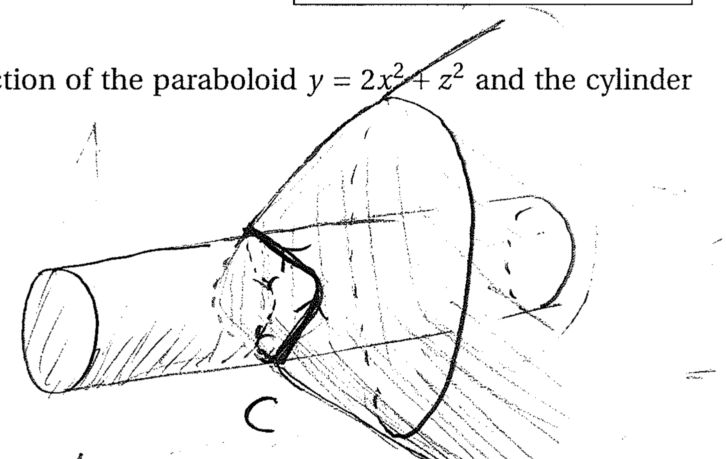
$$= \int_0^1 (t^2 - t, t) \cdot (2t, 1)\, dt = \int_0^1 2t^3 - 2t^2 + t\, dt$$

$$= \frac{2}{4}t^4 - \frac{2}{3}t^3 + \frac{1}{2}t^2 \Big|_{t=0}^{1} = \frac{1}{2} - \frac{2}{3} + \frac{1}{2} = 1/3$$

$\int_C \mathbf{F} \cdot d\mathbf{r} =$ $1/3$

6. Find a parameterization $\mathbf{r}(t)$ of the curve of intersection of the paraboloid $y = 2x^2 + z^2$ and the cylinder $x^2 + z^2 = 1$. **(3 points)**

In xz plane, pts of C are on the unit circle. So take $x = \cos(t)$ and $z = \sin(t)$



Then $y = 2x^2 + z^2 = 2\cos^2 t + \sin^2 t = 1 + \cos^2 t$

So we get

$\mathbf{r}(t) = \langle \cos t, 1 + \cos^2 t, \sin t \rangle$

7. (a) Consider the vector field $\mathbf{F}(x,y) = \langle y^2, 1+2xy \rangle$ on $\mathbb{R}^2$. Show that $\mathbf{F}$ is conservative by finding a function $f(x,y)$ where $\mathbf{F} = \nabla f$. **(3 points)**

Want $\frac{\partial f}{\partial x} = y^2 \Rightarrow f = \int y^2\,dx = xy^2 + C(y)$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy^2 + C(y)\right) = 2xy + \frac{\partial C}{\partial y} \overset{\text{Goal}}{=} 1 + 2xy.$$

So $\frac{\partial C}{\partial y} = 1 \Rightarrow C = y$ (+ const). So we get

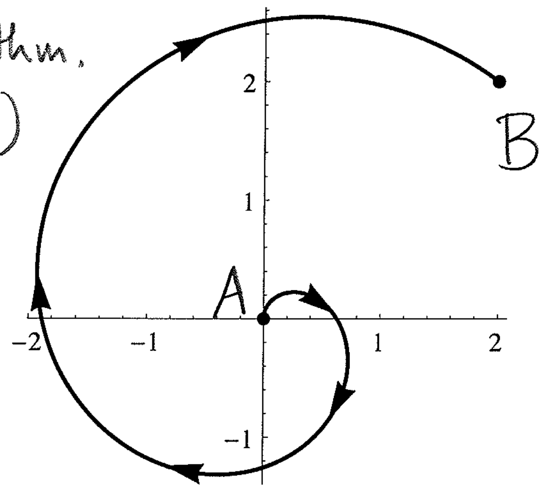
$\boxed{f(x,y) = xy^2 + y}$

(b) For the curve C shown at right, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(2 points)**

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{by fund. thm.}}{=} f(B) - f(A)$$

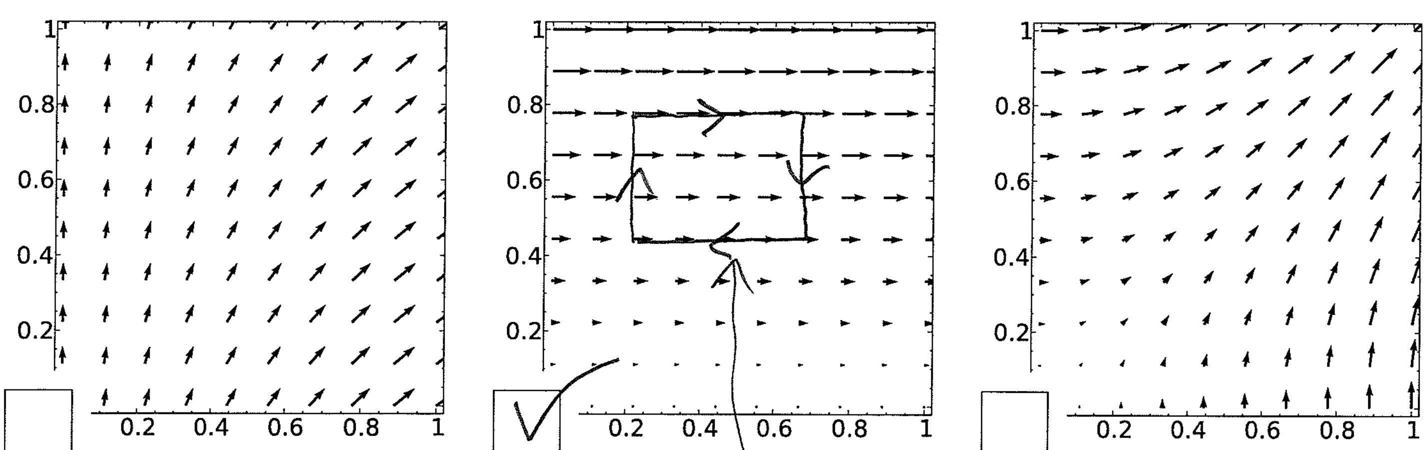
$$= f(2,2) - f(0,0)$$

$$= (8+2) - (0) = 10$$



$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = 10}$

(c) Exactly one of the vector fields below is *not* conservative. Mark the box of the non-conservative vector field. **(2 point)**



$\int_C \vec{F}\cdot d\vec{r} > 0$

8. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by the contour diagram at right, as well as the curve C, the point A, and the vectors $\mathbf{u}$ and $\mathbf{v}$ indicated. For each part, circle the best answer. **(1 point each)**

(a) The sign of $D_{\mathbf{u}} f(A)$.

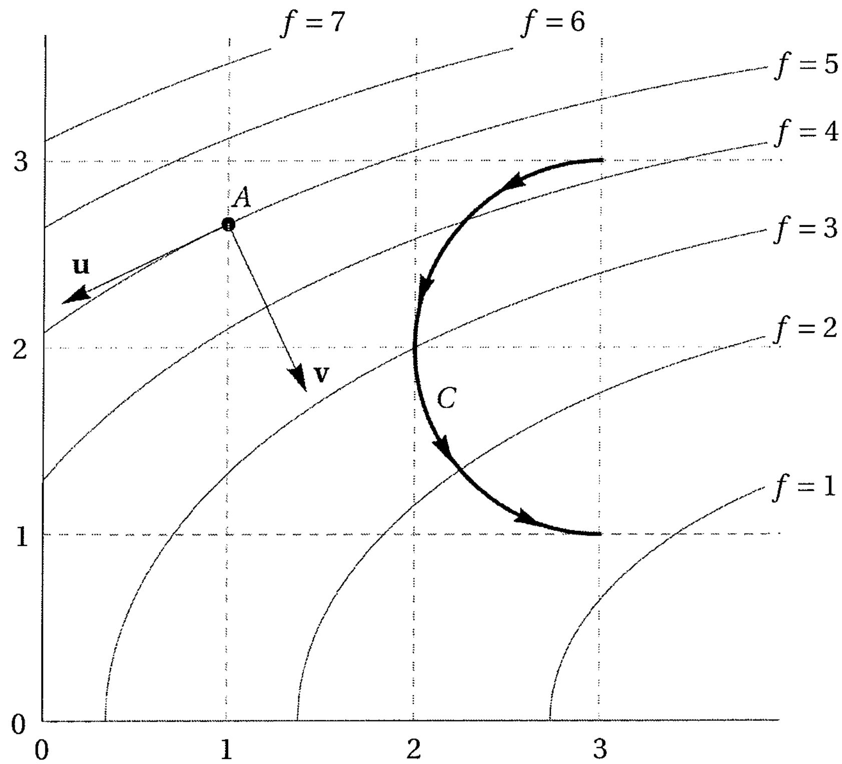
negative	zero	positive

(b) $\nabla f(A) = \mathbf{v}$.

true	false

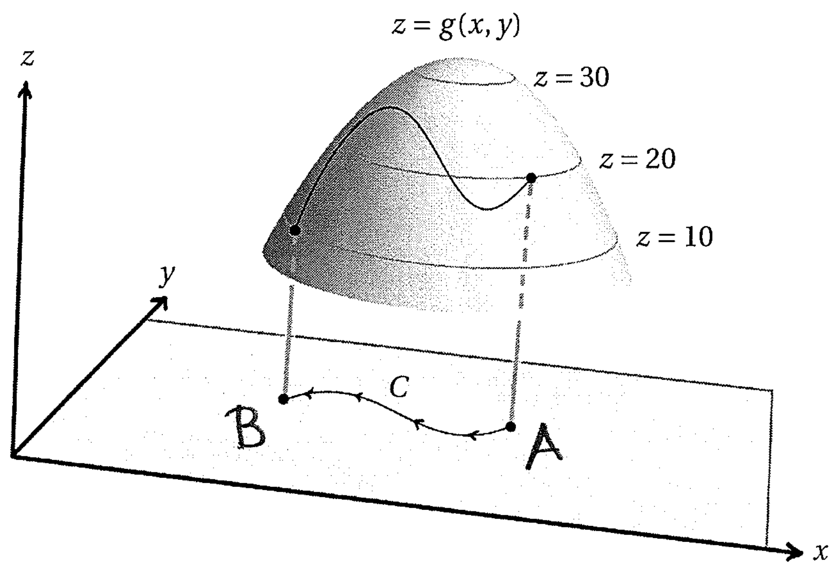
(c) The value of $\int_C f(x,y)\, ds$.

−9	−6	−3	0	3	6	9



9. Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is shown at right, and let C be the indicated curve in the xy-plane. Evaluate the line integral $\int_C \nabla g \cdot d\mathbf{r}$. **(2 points)**

$$\int_C \nabla g \cdot d\vec{r} = g(B) - g(A)$$
$$= 10 - 20 = -10$$



$\int_C \nabla g \cdot d\mathbf{r} = -10$