Math 351 - Elementary Topology
Friday, November 9 $**$ *Exam 2 Review Problems*

1. Give an example of subspaces $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, for some n, together with a continuous bijection $f : A \longrightarrow B$ which is *not* a homeomorphism.

2. Show that if $f : X \longrightarrow Y$ is a homeomorphism and $A \subseteq X$, then $\text{Int}\big(f(A)\big) = f(\text{Int}\, A)$.

3. Let $f : X \longrightarrow Y$ be an embedding.

   (a) Prove or disprove: If Y is Hausdorff, so is X.

   (b) Prove or disprove: If X is Hausdorff, so is Y.

4. Show that if $A \subseteq X$ is closed and $B \subseteq Y$ is also closed, then $A \times B \subseteq X \times Y$ is closed. Use **only** the definition of the product topology. In other words, you may *not* use that $\overline{A \times B} = \overline{A} \times \overline{B}$.

5. Let (x_n) and (y_n) be sequences in the spaces X and Y, respectively. Show that $x_n \to x$ and $y_n \to y$ if and only if $(x_n, y_n) \to (x, y)$ in $X \times Y$.

6. Let $X = \mathbb{R}_\ell \times \mathbb{R}$ and let $L \subseteq X$ be a line. Describe the topology on L inherited from X. Hint: the answer depends on the slope of L.

7. Let $X \times Y$ be partitioned into the subsets $X \times \{y\}$, one partition for each $y \in Y$. Show that the resulting quotient $(X \times Y)^*$ is homeomorphic to Y.

8. Give an example of a quotient map $q : X \twoheadrightarrow Y$ such that q is *not* an open map.

9. Let $Z \subseteq \mathbb{R}^2$ be the union of the two coordinate axes. Define $q : \mathbb{R}^2 \twoheadrightarrow Z$ by

$$q(x, y) = \begin{cases} (x, 0) & x \neq 0 \\ (0, y) & x = 0. \end{cases}$$

   (a) Show that q is *not* continuous if Z is given the subspace topology.

   (b) Describe the resulting quotient topology on Z. What would be a basis for this topology? Is it Hausdorff?

10. Show that a hexagon with opposite edges glued together with a flip yields $\mathbb{RP}^2$.

Solutions.

1. There are many possibilities, but one example that was mentioned in class is $A = [0,1] \cup (2,3]$ and $B = [0,2]$, with the continuous bijection $f : A \longrightarrow B$ defined by

$$f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1 \\ x - 1 & \text{if} \quad 2 < x \le 3. \end{cases}$$

The function f is clearly a bijection (an inverse is $g : B \longrightarrow A$ defined by $g(y) = y$ if $0 \le y \le 1$ aand $g(y) = y + 1$ if $1 < y \le 2$.) Also, f is continuous by the glueing lemma because its restrictions to the closed subsets $[0,1]$ and $(2,3]$ are continuous. However, f is not a homeomorphism because the subset $(2,3]$ is closed in A, whereas $f((2,3]) = (1,2]$ is not closed in $B = [0,2]$.

2. Since f is a homeomorphism $f(\text{Int}(A))$ is open in Y. Also, since $\text{Int}(A) \subseteq A$, it follows that $f(\text{Int}(A)) \subseteq f(A)$. Since $\text{Int}(f(A))$ is the largest open subset in $f(A)$, it follows that

$$f(\text{Int}(A)) \subseteq \text{Int}(f(A)).$$

It remains to show the other inclusion. Let us write $V = \text{Int}(f(A))$ and let $y \in V \subseteq f(A)$. Then we can write $y = f(x)$ for some $x \in A$. We must show that $x \in \text{Int}(A)$. Since $y = f(x)$ is in V, it follows that x is in the set $U = f^{-1}(V)$. Since f is continuous, U is open. Also, since $V \subseteq f(A)$, it follows that

$$U = f^{-1}(V) \subseteq f^{-1}(f(A)) = A.$$

Note that we have used that f is injective to get the last equality. We now have $x \in U \subseteq A$. Since U is open, this implies that $x \in \text{Int}(A)$. Thus $y = f(x) \in f(\text{Int}(A))$. We have demonstrated that

$$\text{Int}(f(A)) \subseteq f(\text{Int}(A)).$$

3. (a) This is true. Let x_1 and x_2 be distinct points in X. The embedding f is injective, so $f(x_1)$ and $f(x_2)$ are distinct points in Y. Let V_1 and V_2 be disjoint neighborhoods of these points in Y. Then $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are disjoint neighborhoods of x_1 and x_2 in X, so X is Hausdorff.

(b) This is false. Let X be any Hausdorff space, like $X = (0,1)$, for example. Let W be any nonHausdorff space, like $W = \mathbb{R}/(0,\infty)$. Then take Y to be the disjoint union $Y = X \amalg W$ and let $f : X \longrightarrow Y$ be the inclusion $f = i_1$. The inclusion a space into the disjoint union with another space is always an embedding. But Y is not Hausdorff because the points 0 and $\overline{1}$ in $W \subseteq Y$ do not have disjoint neighborhoods.

4. Let $A \subseteq X$ and $B \subseteq Y$ be closed. Then the complements $U = X \setminus A$ and $V = Y \setminus B$ are open. We wish to show that $A \times B$ is closed in $X \times Y$, which means that the complement is open. The complement is

$$(X \times Y) \setminus (A \times B) = (U \times Y) \cup (X \times V).$$

The two sets on the right are basis elements in the product topology, so their union is open. It follows that $A \times B$ is closed.

5. ($\Rightarrow$) Assume $x_n \to x$ and $y_n \to y$. Let W be a neighborhood of (x, y) in $X \times Y$. Then there is a basic neighborhood

$$(x, y) \in U \times V \subseteq W.$$

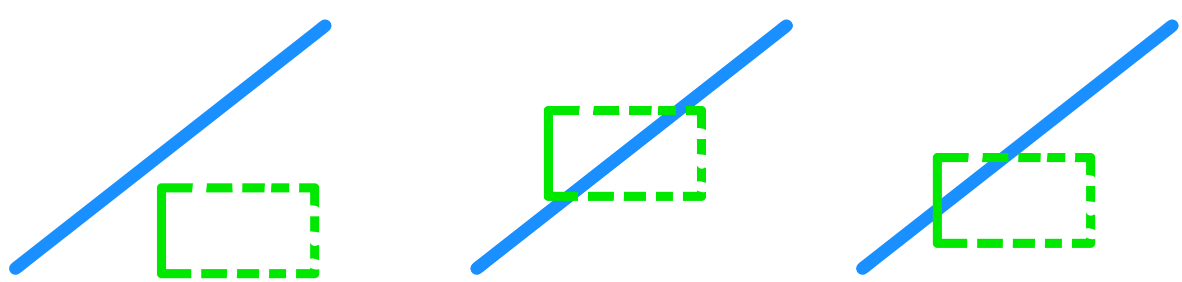
Since $x_n \to x$ and $x \in U$, some tail of the sequence (x_n) is in U. Suppose $\{x_n \mid n > M\} \subseteq U$. Similarly, $y_n \to y$ and $y \in V$, so a tail of this sequence is in V. Suppose $\{y_n \mid n > N\} \subseteq V$. Then if $n > K = \max\{M, N\}$, it follows that $(x_n, y_n) \in U \times V \subseteq W$. In other words, we have shown that a tail of the sequence (x_n, y_n) is in W, so $(x_n, y_n) \to (x, y)$.
($\Leftarrow$) Recall that the projections $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ are continuous. Recall also that continuous functions preserve convergence of sequences. So if $(x_n, y_n) \to (x, y)$ it follows that

$$(x_n) = \pi_1(x_n, y) \to \pi_1(x, y) = x$$

and similarly

$$(y_n) = \pi_2(x_n, y) \to \pi_2(x, y) = y.$$

6. Suppose first that the line L is a vertical line. A basic open set in $\mathbb{R}_\ell \times \mathbb{R}$ is of the form $[a, b) \times (c, d)$. Intersecting this basic open set with a vertical line $x = e$ will give either an empty set if $e \notin [a, b)$ or an interval $\{e\} \times (c, d)$ if $e \in [a, b)$. It follows that the induced topology on this vertical line is the *standard* topology.
Suppose now that the line L is not vertical. Then the intersection of a basic open as described above with the line L will result in either (1) an empty set or (2) an open interval on the line or (3) a half-open interval on the line. See the figures below.



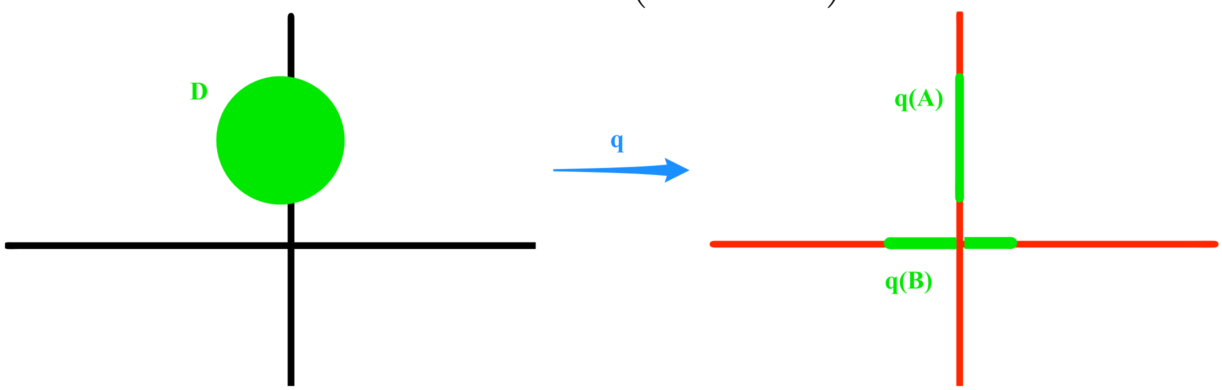
It follows that the induced topology on the line L is the lower limit topology.

7. Since there is one partition for each $y \in Y$, it it clear that the set $(X \times Y)^*$ is in bijection with Y and that the quotient map $q : X \times Y \longrightarrow (X \times Y)^*$ can be modeled as the projection $X \times Y \longrightarrow Y$. It only remains to verify that the topology agrees with the topology of Y. A subset $U \subseteq (X \times Y)^* = Y$ is open if and only if $q^{-1}(U) = X \times U$ is open in $X \times Y$. The projection map $\pi_2 : X \times Y \longrightarrow Y$ is continuous and open, so it follows that $U \subseteq Y$ is open if and only if $X \times U = \pi_2^{-1}(U) \subseteq X \times Y$ is open.

8. Consider the quotient $q : \mathbb{R} \longrightarrow \mathbb{R}/[0,1]$. Then $(0,1) \subseteq \mathbb{R}$ is open, but $q(0,1)$ is the collapsed point $\overline{0}$ in the quotient. The set $q^{-1}(\overline{0}) = [0,1] \subseteq \mathbb{R}$ is closed but not open, so $\overline{0}$ is closed and not open in the quotient.

9. (a) The subset $U = \{0\} \times (1,2) \subseteq Z$ is open in the subspace topology, but the preimage $q^{-1}(U) = \{0\} \times (1,2) \subseteq \mathbb{R}^2$ is not open (it is nonempty but does not contain *any* open discs).

(b) As is shown in the textbook, applying the map q to the basis for $\mathbb{R}^2$ will provide a basis for the quotient topology on Z.
Let D be an open disc in $\mathbb{R}^2$ that does not meet the y-axis. Then $q(Z)$ is simply an open interval on the x-axis in Z that does not contain the origin.
Now let D be an open disc in $\mathbb{R}^2$ that intersects the y-axis nontrivially. Then we can write $D = A \cup B$, where $A = D \cap \big(\{0\} \times \mathbb{R}\big)$ and B is the complementary piece $B = D \setminus A$. Then $q(A) = A$, but $q(B)$ is the union of intervals $\Big((a,0) \cup (0,b)\Big) \times \{0\}$, where $a < 0$ and $b > 0$.



So any point on the x-axis (including the origin) will have neighborhoods as in the subspace topology, but neighborhoods of points in the y-axis necessarily include positive and negative intervals on the x-axis.
The space Z with the quotient topology is *not* Hausdorff because no two points on the y-axis can have disjoint neighborhoods.

10. No cutting-and-pasting is needed for this problem. Recall that the projective plane was originally defined as the quotient of the square, in which opposite sides are identified with a flip. We saw that this was the same as a disc with the two sides of the boundary circle identified with a flip. The hexagon description agrees with both of these, without any cutting-and-pasting.