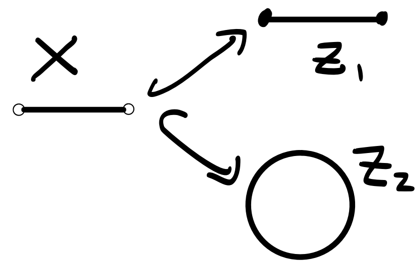
Group: ______________ ______________ ______________ Name: _______________

Math 351 - Elementary Topology
Wednesday, December 5 $**$ *One-point Compactification*

A **compactification** of a space X is a compact space Z together with an embedding $X \hookrightarrow Z$ such that X is dense in Z. Pictured to the right are two compactifications of an open interval.



Given any space X, define a new space $X_+ = X \cup \{\infty\}$ as follows:

- every open subset $U \subseteq X$ is considered open in X_+
- The neighborhoods V of the new point ∞ are the subsets such that $X \setminus V$ is *compact*.

It can be shown that if X is Hausdorff then this defines a topology on X_+, and it is clear that X is a subspace of X_+.

1. Assume the above defines a topology on X_+. Show that X_+ is compact.
2. What space is $\mathbb{R}_+$? What are $\mathbb{R}^2_+$ and $\mathbb{R}^n_+$?
3. What is $\Big[(0,1) \cup (2,3)\Big]_+$?
4. Assume X is **not** compact. Show that X is dense in X_+. What is another description of X_+ if X is already compact?

Solutions.

1. Let $\mathcal{U}$ be an open cover of X_+. Then at least one of the open sets in $\mathcal{U}$ contains the added point ∞. Let $\infty \in V \in \mathcal{U}$. We may assume that V is not all of X_+, since if $V = X_+$ then $\{V\} \subseteq \mathcal{U}$ is a finite subcover. Now $\mathcal{U} \setminus \{V\}$ must cover $X_+ \setminus V$. By the definition of the topology on X_+, the set $X_+ \setminus V$ is compact, so $\mathcal{U} \setminus \{V\}$ has a finite subcover $\mathcal{V}$ (as a cover of $X_+ \setminus V$). It follows that $\mathcal{V} \cup \{V\}$ is a finite subcover of X_+, and X_+ is compact.
2. These spaces are S^1, S^2, and S^n, respectively. This can be seen by using the stereographic projection, which identifies $S^n \setminus \{P\}$ with $\mathbb{R}^n$ for any $P \in S^n$. Note that every open neighborhood U of P must have compact complement since $S^n \setminus U$ is a closed subset of the compact space S^n.
3. This space can be identified with $S^1 \vee S^1$. The point ∞ corresponds to the common basepoint of the two circles.

4. Suppose X is not compact. Then the point $\{\infty\}$ is not open, which means that X is not closed in X_+. It follows that $X_+ = X \cup \{\infty\}$ is the smallest closed set containing X, so that X is dense in X_+.

   If we assume, on the other hand, that X is compact, then $\{\infty\}$ is both closed and open in X_+. In other words, $X_+ = X \amalg \{\infty\}$ is a disjoint union.