Group: ____________ ____________ ____________ Name: ____________

Math 351 - Elementary Topology
Wednesday, September 26 $**$ *The subspace topology*

The problems below concern the subspace topology. Recall that if X is a topological space and $A \subseteq X$ is a subset, we define the **subspace topology** on A by specifying that a subset

$$V \subset A \quad \text{is open} \Leftrightarrow \quad V = U \cap A \quad \text{for some open set} \quad U \subset X.$$

Make sure to justify all of your answers.

1. (1 **point**) Show that if X is Hausdorff and $A \subseteq X$, then A is also Hausdorff if it is given the subspace topology.

2. (**2 points**) Let

$$A \subseteq X \qquad \text{and} \qquad B \subseteq A.$$

Then A can be considered as a subspace of X and B can be considered as a subspace of A. But B can *also* be considered as a subspace of X. **Show that** the two resulting subspace topologies on B (one coming from A and the other from X) are in fact the same topology.

Write your answer(s) on the rest of this sheet (and back).

Solutions.

1. Suppose X is Hausdorff and let $A \subseteq X$. Let $a \neq b$ be distinct points of A. Since $A \subseteq X$, then a and b are also distinct points in X. Since X is Hausdorff, there are disjoint neighborhoods (in X) U and V of a and b, respectively. But then the sets

$$U_A = U \cap A, \qquad V_A = V \cap A$$

are disjoint subsets of A that contain a and b, respectively. Finally, they are open by the definition of the subspace topology on A. So A is Hausdorff.

2. We will write B_A and B_X for the set B equipped with the topologies coming from A and X, respectively. Suppose that $U \subseteq B_A$ is open. This means that $U = V \cap B$ for some open $V \subseteq A$. But by the definition of the topology on A, this means that $V = W \cap A$ for some open $W \subseteq X$. Then

$$U = V \cap B = (W \cap A) \cap B = W \cap (A \cap B) = W \cap B,$$

so U is open in B_X.
On the other hand, if $U \subseteq B_X$ is open, then $U = W \cap B$ for some open $W \subseteq X$. Then $V = W \cap A$ is open in A and

$$U = W \cap B = W \cap (A \cap B) = (W \cap A) \cap B = V \cap B$$

is open in B_A.