Group: ____________ ____________ ____________ Name: ____________

Math 351 - Elementary Topology
Wednesday, October 3 ∗∗ *Continuous functions*

The problems below concern continuous functions. Recall that $f : X \longrightarrow Y$ is continuous if for every open set $U \subset Y$, the preimage $f^{-1}(U)$ is open in X. Make sure to justify all of your answers.

1. (2 **points**) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x & x \geq 0 \\ x-1 & x < 0. \end{cases}$$

Show that f is continuous when considered as a function

$$f : \mathbb{R}_{\ell\ell} \longrightarrow \mathbb{R}.$$

($\mathbb{R}_{\ell\ell}$ is $\mathbb{R}$ with the "lower-limit", or half-open, topology.) Feel free to use the result from class that f is continuous if and only if it is continuous at every $c \in \mathbb{R}_{\ell\ell}$.

2. (**2 points**) Let $f, g : X \longrightarrow Y$ be continuous, and suppose that Y is Hausdorff. Show that if $D \subset X$ is *dense* in X and $f(d) = g(d)$ for all $d \in D$, then necessarily $f(x) = g(x)$ for all $x \in X$. In other words, show that if f and g agree on a dense subset, then they agree everywhere.

Write your answer(s) on the rest of this sheet (and back).

Solutions.

1. Let's first consider continuity at some point other than 0.
Case I: $c > 0$. Then let U be any neighborhood of $f(c) = c$. We may assume that U is a basis element (a, b), and by shrinking the neighborhood if necessary we may assume $a > 0$. Then $f^{-1}((a, b)) = (a, b)$, which is open in $\mathbb{R}_{\ell\ell}$.
Case II: $c < 0$. We may consider an interval (a, b) containing $f(c) = c - 1$, and we may assume $b < -1$. Then $f^{-1}((a, b)) = (a + 1, b + 1)$, which is again open in $\mathbb{R}_{\ell\ell}$.
Case III: $c = 0$. Let (a, b) be a neighborhood of $f(0) = 0$. Again, we are free to shrink the neighborhood, so we may assume $a > -1$. Then $f^{-1}((a, b)) = [0, b)$, which is open in $\mathbb{R}_{\ell\ell}$.

2. Let $D \subseteq X$ be a dense subset on which the functions f and g agree and let $x \in X$ be any point.
Let us assume, for a contradiction, that $f(x) \neq g(x)$ in Y. Since Y is Hausdorff, this means we can find disjoint neighborhoods U and V of $f(x)$ and $g(x)$, respectively. Since f is continuous, $f^{-1}(U)$ is a neighborhood of x in X. Similarly, $g^{-1}(V)$ is a neighborhood of x in X. It follows that their intersection

$$W = f^{-1}(U) \cap g^{-1}(V)$$

is also a neighborhood of x in X. Since D is dense, it follows that $D \cap W$ is nonempty. Let $d \in D \cap W$. Since $d \in D$, we have $f(d) = g(d)$. On the other hand, $f(d) \in U$ and $g(d) \in V$. Since $f(d) = g(d)$, this contradicts the fact that U and V are disjoint. $\lightning$