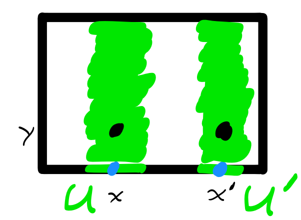
Group: ____________ ____________ ____________ Name: ____________

Math 351 - Elementary Topology
Wednesday, October 24 $**$ *Products and the Hausdorff condition*

1. Show that if X and Y are Hausdorff spaces, then so is their product $X \times Y$.
2. Show that X is Hausdorff if and only if the diagonal subset
$$\Delta(X) = \{(x, y) \in X^2 \mid x = y\} \subseteq X \times X$$
is closed.
3. Let $f, g : X \longrightarrow Y$ be continuous, and suppose that Y is Hausdorff. Show that if $D \subset X$ is *dense* in X and $f(d) = g(d)$ for all $d \in D$, then necessarily $f(x) = g(x)$ for all $x \in X$. Hint: This was on a previous worksheet, but it now follows easily from problem 2 above.

Write your answer(s) on the rest of this sheet (and back).

Solutions.



1. Let (x, y) and (x', y') be distinct points in $X \times Y$. This means that either $x \neq x'$ or $y \neq y'$ (or both). Without loss of generality, we assume $x \neq x'$. Then, since X is Hausdorff, this means we can find disjoint neighborhoods U and U' of x and x' in X. It follows that $U \times Y$ and $U' \times Y$ are disjoint neighborhoods of (x, y) and (x', y').

2. $(\Rightarrow)$ Assume X is Hausdorff. We will show that $X \times X \setminus \Delta(X)$ is open. Let $(x, y) \notin \Delta(X)$. This means that $x \neq y$. So there must be disjoint neighborhoods U and V of y. Then $U \times V$ is a neighborhood of (x, y) in $X \times X$, and it does not meet $\Delta(X)$, since if $(z, z) \in \Delta(X) \cap (U \times V)$, this would mean that $z \in U \cap V$, which would contradict that $U \cap V = \emptyset$.

$(\Leftarrow)$ Assume $\Delta(X)$ is closed. Let $x \neq y$ be distinct points in X. Then (x, y) lies in the open set $(X \times X) \setminus \Delta(X)$. By the definition of the product topology, this means that we can find a basic open set $U \times V$ with
$$(x, y) \in U \times V \subseteq (X \times X) \setminus \Delta(X).$$
The fact that $U \times V$ misses the diagonal says precisely that $U \cap V$ is empty.

3. Let f and g be as described in the statement of the problem. Consider $\Delta(Y) \subseteq Y \times Y$. Since Y is Hausdorff, by problem 2 we know this diagonal subset is closed. The function $(f, g) : X \longrightarrow Y \times Y$ is continuous by the universal property of products. We conclude that
$$(f, g)^{-1}(\Delta(Y)) = \{x \in X \mid f(x) = g(x)\}$$
is closed in X. Since this closed set contains the dense set D, this closed set must be all of X.