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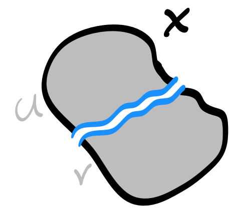
Math 351 - Elementary Topology
Wednesday, November 28 $**$ *Connectedness*

The following are equivalent ways of stating that a space X is **connected**



- The only nonempty closed and open subset $U \subseteq X$ is X itself.
- If $X = U \amalg V$ with U and V both open, then either U or V is empty.

Define an equivalence relation $\sim$ on X by $x \sim y$ if there exists a *connected* subset of X that contains both x and y.
The equivalence class $\overline{x}$ of $x \in X$ is called the "connected component" of x in X.

1. Show that the relation defined above is transitive.
2. Show that a "connected component" is in fact connected.
3. Show that $\mathbb{R}_{fc}$ has only one component (in other words, show it is a connected space).
4. Find an example of a space X and a connected component $C \subset X$ such that C is *not* open in X. (The components are always closed, however.)

Solutions.

1. Suppose that $x \sim y$ and $y \sim z$. Then there is some connected subset $C \subseteq X$ with $x, y \in C$ and also a connected subset $D \subseteq X$ such that $y, z \in D$. Then $y \in C \cap D$, so in particular $C \cap D$ is nonempty. By a result from class, we conclude that the union $C \cup D$ is also connected. Since x and z are in $C \cup D$, it follows that $x \sim z$.
2. Again, the idea is to take a union. Let $x \in X$. We wish to show the component $\overline{x}$ is connected. By definition, $y \in \overline{x}$ if and only if $y \in C$ for some connected subset $C \subset X$ containing x. This means that we can write
$$\overline{x} = \bigcup_{\substack{C \ni x, \\ C \text{ connected}}} C.$$
The intersection of all of these connected sets C is nonempty since x lies in each C. By the same result from class, we conclude that the union of all of these C's, in other words the component $\overline{x}$, is connected.

3. Suppose that $W \subseteq \mathbb{R}_{fc}$ is open and closed and nonempty. Since W is open and nonempty, the complement $\mathbb{R} - W$ is finite. On the other hand, W is closed and so must either be finite or all of $\mathbb{R}$. W cannot be finite since its complement is finite. So the only possibility is $W = \mathbb{R}$.

4. An example is $X = \mathbb{Q}$. Any subset $A \subseteq \mathbb{Q}$ containing at least two points is disconnected. The reason is that if z is any irrational number between those two chosen points, then $(-\infty, z) \cap A$ and $(z, \infty) \cap A$ give a separation of A. It follows that the connected components in $\mathbb{Q}$ are the singletons.