Math 351 - Elementary Topology
Monday, October 8 $\ast\ast$ *Exam 1 Review Problems*

1. Say $U \subseteq \mathbb{R}$ is open if either it is finite or $U = \mathbb{R}$. Why is this not a topology?
2. Let $X = \{a, b\}$.
   (a) If X is equipped with the trivial topology, which functions $f : X \longrightarrow \mathbb{R}$ are continuous? What about functions $g : \mathbb{R} \longrightarrow X$?
   (b) If X is equipped with the topology $\{\emptyset, \{a\}, X\}$, which functions $f : X \longrightarrow \mathbb{R}$ are continuous? What about functions $g : \mathbb{R} \longrightarrow X$?
   (c) If X is equipped with the discrete topology, which functions $f : X \longrightarrow \mathbb{R}$ are continuous? What about functions $g : \mathbb{R} \longrightarrow X$?
3. Given an example of a topology on $\mathbb{R}$ (one we have discussed) that is *not* Hausdorff.
4. Show that if $A \subset X$, then $\partial A = \emptyset$ if and only if A is both open and closed in X.
5. Give an example of a space X and an open subset A such that $\text{Int}(\overline{A}) \neq A$.
6. Let $A \subset X$ be a subspace. Show that $C \subset A$ is closed if and only if $C = D \cap A$ for some closed subset $D \subset X$.
7. Show that the addition function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined by $f(x, y) = x + y$, is continuous.
8. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous only at 0 (in the usual topology). Hint: Define f piecewise, using the rationals and irrationals as the two pieces.

Solutions.

1. This fails the union axiom. Any singleton set would be open. The set $\mathbb{N}$ is a union of singleton sets and so should also be open, but it is not. So this is not a topology.

2. $X = \{a, b\}$.
a) X has the trivial topology: Only constant functions $X \longrightarrow \mathbb{R}$ are continuous, since if $f(a) \neq f(b)$, then $U = \mathbb{R} \setminus \{f(b)\}$ is open in $\mathbb{R}$, but $f^{-1}(U) = \{a\}$ would not be open in X. On the other hand, every function $\mathbb{R} \longrightarrow X$ is continuous by homework problem 4.1(b).

b) X has the topology $\{\emptyset, \{a\}, X\}$: This is the same as part a). If $f : X \longrightarrow \mathbb{R}$ is not constant, then $U = \mathbb{R} \setminus \{f(a)\}$ is open in $\mathbb{R}$, but $f^{-1}(U) = \{b\}$ is not open in X. For a function $f : \mathbb{R} \longrightarrow X$, the only requirement for it to be continuous is that $f^{-1}(a)$ is open in $\mathbb{R}$. So for each open set $U \subseteq \mathbb{R}$, there is a continuous function $f : \mathbb{R} \longrightarrow X$ defined by

$$f(x) = \begin{cases} a & x \in U \\ b & x \notin U. \end{cases}$$

c) X has the discrete topology: By homework problem 4.1(a), every function $X \longrightarrow \mathbb{R}$ is continuous. The only continuous functions $g : \mathbb{R} \longrightarrow X$ are the constant functions. To see this, note that $\{a\}$ is both open and closed in X. So $g^{-1}(\{a\})$ must also be closed and open in $\mathbb{R}$. But the only closed and open sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$ (this was the challenge problem on HW1). If $g^{-1}(\{a\}) = \emptyset$, this means g is constant at b, and if $g^{-1}(\{a\}) = \mathbb{R}$, this means g is constant at a.

3. The cofinite topology on $\mathbb{R}$ is not Hausdorff: if U is a neighborhood of 0 and V is a neighborhood of 1, then the complements $A = \mathbb{R} \setminus U$ and $B = \mathbb{R} \setminus V$ are finite. But then the complement of $U \cap V$ is $A \cup B$, which is also finite. since $U \cap V$ has finite complement, it is in particular nonempty. This shows that $\mathbb{R}_{cf}$ is not Hausdorff.

4. Let $A \subset X$.
($\Rightarrow$) Suppose $\partial A = \emptyset$. Since $\partial A = \overline{A} \setminus \text{Int}(A)$, this means that $\overline{A} = \text{Int}(A)$. But we always have the inclusions

$$\text{Int}(A) \subset A \subset \overline{A},$$

so combining this with

$$A \subset \overline{A} = \text{Int}(A) \qquad \text{and} \qquad \overline{A} = \text{Int}(A) \subset A$$

gives the identifications

$$A = \text{Int}(A) \qquad \text{and} \qquad A = \overline{A}.$$

In other words, A is closed and open.
($\Leftarrow$) If A is closed and open, then $A = \text{Int}(A)$ and $A = \overline{A}$, so

$$\partial(A) = \overline{A} \setminus \text{Int}(A) = A \setminus A = \emptyset.$$

5. For this problem, it is enough to find a space X with an open dense set A, since then the closure of A will be X and therefore open. An example would be any set X (at least two points) with a particular point topology. Then let A be any set containing the particular point but not equal to all of X.

6. Let $A \subset X$ be a subspace.
($\Rightarrow$) Assume $C \subset A$ is closed. This means that if we let $U = A \setminus C$, then $U = V \cap A$ for some open $V \subset X$. Then $D = X \setminus V$ is closed in X and

$$C = A \setminus U = A \setminus (V \cap A) = A \setminus ((X \setminus D) \cap A) = A \setminus (A \setminus (D \cap A)) = D \cap A.$$

($\Leftarrow$) Assume $C = D \cap A$ for some closed $D \subset X$. Then $V = X \setminus D$ is open in X and

$$A \setminus C = A \setminus (D \cap A) = A \setminus ((X \setminus V) \cap A) = A \setminus (A \setminus (V \cap A)) = V \cap A.$$

This means that $A \setminus C$ is open, so C must be closed.

7. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the addition function given by $f(x,y) = x + y$. Let $(a,b) \subseteq \mathbb{R}$ be a basic open set. We need to show that $f^{-1}((a,b))$ is open in $\mathbb{R}^2$. This is the intersection of the two diagonal half-planes

$$U = \{(x,y) \mid x + y < b\} \qquad \text{and} \qquad V = \{(x,y) \mid x + y > a\}.$$

We already saw earlier in the course that half-planes like these are open in $\mathbb{R}^2$, but here is the argument for V.
Suppose $(x,y) \in V$. By translating the in the x-direction by a quantity of $-a$ and then rotating by 45° clockwise, we find that the distance of the point (x,y) from the line $x + y = a$ is

$$\frac{\sqrt{2}}{2}(x + y - a).$$

So we may take a ball with center (x,y) and radius $\frac{\sqrt{2}}{2}(x + y - a)$ as a neighborhood of (x,y) in V.

8. Here is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous *only* at $x = 0$. We define

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then if $U = (a,b)$ is a neighborhood of $f(0) = 0$, we have $0 \in V = (a,b) \subseteq f^{-1}(U)$. This shows f is continuous at 0.
On the other hand, let $c \neq 0$. For simplicity, we assume $c > 0$.
Case I: ($c \in \mathbb{Q}$): then $f(c) = c$ and $U = (0, 2c)$ is a neighborhood of $f(c)$, but

$$f^{-1}(U) = (0, 2c) \cap \mathbb{Q}$$

does not contain any neighborhood of c.
Case II: ($c \notin \mathbb{Q}$): then $f(c) = 0$ and $U = (-c/2, c/2)$ is a neighborhood of 0. But $f^{-1}(U)$ does not contain any neighborhood of c since it does not contain any points from $(c/2, c) \cap \mathbb{Q}$.