wa1-1

(1a) $$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$$

$$f(x) = (1+x)^\alpha$$

$$f^{(j)}(x) = \alpha(\alpha-1)\cdots(\alpha-(j-1))\,(1+x)^{\alpha-j}$$

$$f^{(j)}(0) = \alpha(\alpha-1)\cdots(\alpha-(j-1))$$

Binomial Coef: $$\frac{\alpha(\alpha-1)\cdots(\alpha-(j-1))}{j!} = \frac{\alpha!}{(\alpha-j)!\,j!}$$
$$= \binom{\alpha}{j}$$

So
$$T_n(x) = \sum_{j=0}^{n} \binom{\alpha}{j} x^j$$

(1b) $$R_n(x) = \frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du$$

since
$$f^{(n+1)}(u) = \alpha(\alpha-1)\cdots(\alpha-n)(1+u)^{\alpha-(n+1)}$$

we get

$$R_n(x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\int_0^x (x-u)^n (1+u)^{\alpha-(n+1)}\,du$$

If $\alpha<0$ and we restrain ourselves to $[0,x]$

since $(1+u)^{\alpha-j}$ is then monotone decr.

its max is at $u=0$:

$$|R_n(x)| \le \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\int_0^x (x-u)^n\,du$$

$$\le \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\left(\frac{-(x-u)^{n+1}}{n+1}\right)\Big|_0^x$$

$$\le \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}|x|^{n+1}$$

$$= \binom{\alpha}{n+1}|x|^{n+1}.$$

2(a) Take $\alpha = -\frac{1}{2}$ & $x = -\left(\frac{v}{c}\right)^2$.

$$E(v) = T_n(v) + R_n(v)$$

$$T_n(v) = mc^2\left(\sum_{j=0}^{n}\binom{-\frac{1}{2}}{j}\left(-\left(\frac{v}{c}\right)^2\right)^j\right)$$

For part (b):

$$T_0(v) = mc^2\binom{-\frac{1}{2}}{0} = mc^2 = E_0$$

$$T_1(v) = mc^2 + mc^2\binom{-\frac{1}{2}}{1}\left(-\left(\frac{v}{c}\right)^2\right) = mc^2 + \frac{1}{2}mv^2$$

since $\binom{-\frac{1}{2}}{1} = \frac{(-\frac{1}{2})!}{(-\frac{3}{2})!} = -\frac{1}{2}$

wa1-3

$$R_n\left(-\left(\frac{v}{c}\right)^2\right) = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n\right)}{n!}\int_0^{-\left(\frac{v}{c}\right)^2}\left(-\left(\frac{v}{c}\right)^2-u\right)^n$$

$$\times\left(1-u\right)^{-\frac{1}{2}-(n+1)}du$$

not much more one can do here.

(2b) $E_{kin} = E(v) - E_0 = \underbrace{\left(mc^2+\frac{1}{2}mv^2\right)}_{T_1(v)} + R_1(v) - E_0$

we expanded $E(v)$ using $n=1$.

$$E_{kin}(v) = \frac{1}{2}mv^2 + R_1\left(-\left(\frac{v}{c}\right)^2\right)$$

so

$$\left|E_{kin}(v) - \frac{1}{2}mv^2\right| \leq \left|R_1\left(-\left(\frac{v}{c}\right)^2\right)\right|$$

from (1b) $\leq \binom{-\frac{1}{2}}{2}\left(\frac{v}{c}\right)^4$

$$\leq \frac{3}{8}\left(\frac{v}{c}\right)^4.$$

Solve for $\frac{3}{8}\left(\frac{v}{c}\right)^4 \leq 10^{-6}$

$$\left(\frac{v}{c}\right) \leq \left[\frac{8}{3}\times10^{-6}\right]^{1/4} = 0.04 = 4\times10^{-2}$$

$V < 1.2\times10^7$ m/s.

$V_{rocket}^{min} = 1.1\times10^4$ m/s

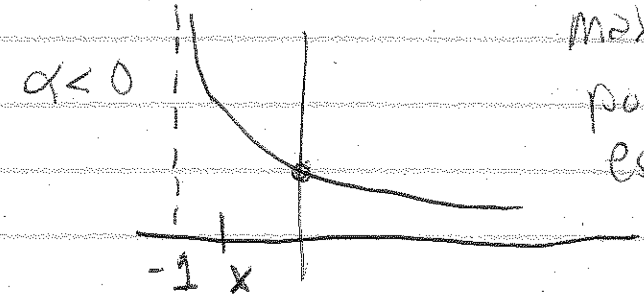
Compare with $\frac{V_{rocket}}{c} = \frac{1.12\times10^4}{3\times10^8} = \frac{1.12}{3}\times10^{-4}$

$= 3.7\times10^{-5}$

which is much smaller.

Remark

Return to (1b): If $x<0$ then the max occurs at that point. The error estimate gets worse as $x \to -1$:



$$|R_n(x)| \leq (1+x)^{-|\alpha|-n-1} |x|^{n+1} \binom{\alpha}{n+1}$$

Applying this to Problem 2, $x = -\left(\frac{v}{c}\right)^2 < 0$

The closer v gets to c the bigger the upper bound and the worse the approximation

$$R_n\left(-\left(\frac{v}{c}\right)^2\right) \leq \left[1-\left(\frac{v}{c}\right)^2\right]^{-\frac{3}{2}-n} \left(\frac{v}{c}\right)^{2(n+1)} \binom{-\frac{1}{2}}{n+1}$$

$$\left| E_{kin}(v) - \frac{1}{2}mv^2 \right| \leq \left[1-\left(\frac{v}{c}\right)^2\right]^{-5/2} \left(\frac{v}{c}\right)^4 \left(\frac{3}{8}\right)$$

$n=1$

This is the correct error bound.