MA 114H Calculus II, Fall 2014
Written Assignment #2
Due **Friday, 14 November 2014**, at the beginning of lecture.

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. Your complete assignment should have your name and section number on each page, be stapled, and be neat and legible. *Unreadable work will receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, careful explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. Your textbook provides examples of how we communicate mathematics.

Students are encouraged to use word-processing software to produce high quality solutions. However, you may find that it is simpler to add graphs and equations using pen or pencil.

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1. (10 points) The goal of this exercise is to derive the error bound for the trapezoidal rule of numerical approximation. If f is a twice differentiable function on $[a,b]$, with a bounded second derivative, and $T_n(f)$ is the n^{th}-trapezoidal approximation, then we want to show that

$$\left| T_n(f) - \int_a^b f(x)\, dx \right| \le \frac{K_2(f)(b-a)^3}{12n^2},$$

where $K_2(f) = \max_{x\in[a,b]} |f''(x)|$.

(a) (2 points) We want to estimate the integral:

$$I(f) = \int_a^b f(x)\, dx.$$

We begin by taking $n = 1$. Write the formula for $T_1(f)$ and for $R_1(f) = T_1(f) - I(f)$.

(b) (2 points) Look at the difference $R_1(f) = T_1(f) - I(f)$. Express this difference as a single integral over $[a,b]$ involving $f'(x)$ and $(x - x_M)$ using integration by parts, where $x_M = \frac{1}{2}(a+b)$ is the midpoint of the interval $[a,b]$. We are looking for a formula of the form

$$R_1(f) = \int_a^b g(x) f'(x)\, dx,$$

so that when you apply the integration by parts method to this, for a certain choice of $g(x)$, you get exactly $R_1(f)$.

(c) (2 points) Use Integration by parts again for the integral obtained above. Show that $R_1(f)$ may be written as

$$R_1(f) = \frac{1}{2}\int_a^b \left\{ \left(\frac{b-a}{2}\right)^2 - (x - x_M)^2 \right\} f''(x)\, dx.$$

You will need to remember that

$$f'(b) - f'(a) = \int_a^b f''(x)\, dx,$$

by the Fundamental Theorem of Calculus.

(d) (2 points) Now consider a partition of $[a, b]$ into n-equal length subintervals. By breaking the integration over $[a, b]$ into n-intervals $[x_{j-1}, x_j]$, $j = 1, \ldots, n$, write $R_n(f) = I(f) - T_n(f)$ as a sum of integrals over each subinterval $[x_{j-1}, x_j]$. We apply the results obtain above to each subinterval noting that now the length of each subinterval is $\frac{(b-a)}{n}$ and that the midpoint of the j^{th}-interval is $x_{M,j} = \frac{(x_{j-1}+x_j)}{2}$.

(e) (2 points) In each term, bound the second derivative by $K_2(f)$, integrate over the interval $[x_{j-1}, x_j]$, sum, and obtain the upper bound on $R_n(f)$ given above.