Sym6 Six weeks of fun! Course Overview

### Part I

# Sym(6): Six weeks of fun!



1 Course Overview

- Goals and expectations
- Overview of course
- 2 First week readings
  - Tuesday
  - Thursday

Sym6 Six weeks of fun! Course Overview Goals and expectations

#### Goals and expectations

- My goal is to make some portion of the literature in algebra and combinatorics more accessible, especially the excellent book by B. Sagan
- This is the "movie version" of the book, where important themes are made clear, but details are freely omitted
- During the first three weeks participants are expected to come with an open mind and to read through the first two chapters.
- During the last three weeks, some volunteers will be expected to be bold and explain to us some facet of Sagan's book

Sym6 Six weeks of fun! Course Overview Overview of course

## Overview of Course

- In the first three weeks we will setup an algebraic and combinatorial scaffolding so that we can work on the high points of the book safely.
- In the last three weeks we will look at the extremely interesting interaction of algebra and combinatorics present in the ordinary representation theory of the symmetric groups.
- First three weeks:
  - Permutations and Tableaux
  - 2 Character theory and Robinson-Schensted
  - 3 Representation theory and the Branching Rule

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### Outline of the reading for Tuesday

- (1.1) Cycle notation, Cycle types, and Partitions
- Conjugacy in Sym(n) and Centralizers
- (1.2) Permutation matrices and representations
- Fixed point formula
- (1.3) Coset representatives
- (1.12) Transversals
- (1.6) Permutation action on Young tabloid
- Exercises: Ch1: #1,2,3,5c,11

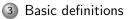
Sym6 Six weeks of fun! First week readings Thursday

### Outline of the reading for Thursday

- (1.6) Tabloids
- (2.1) Tableaux etc.
- Theorem 2.1.12 for permutations, not modules
- (2.3) Row and column stabilizers
- Exercises: Ch2: #1,2,8,10

## Part II

# Week 1A: Permutations



- 4 Conjugacy and centralizers
- 5 Alternate represenations

#### 6 Conclusion

Sym6 Six weeks of fun! Basic definitions Permutations, symmetric group, fixed and moved points

### Permutations

- A **permutation** on a set  $\Omega$  is a one-to-one, onto function from  $\Omega$  to  $\Omega$
- $\bullet\,$  Two permutations on  $\Omega$  can be composed as functions to yield another permutation
- The set of all such functions is called the symmetric group on Ω and denoted Sym(Ω)
- When  $\Omega = \{1, 2, \dots, n\}$  we write  $Sym(\Omega) = Sym(n) = S_n$

• #Sym(n) = n!

If π(x) = x, then x is a fixed point, otherwise it is a moved point.

Sym6 Six weeks of fun! Basic definitions Cycle notation

### Cycle notation

• A cycle  $\pi = (a_1, a_2, \dots, a_n)$  is a special type of permutation

• 
$$\pi(x) = \begin{cases} a_{i+1} & \text{if } x = a_i, 1 \le i < n \\ a_1 & \text{if } x = a_n \\ x & \text{otherwise} \end{cases}$$

- Two cycles are **disjoint** if no point of  $\Omega$  is moved by both
- Cycle notation for a permutation  $\pi$  is a formal product of disjoint cycles whose actual product is  $\pi$

• Example: 
$$\pi = (1, 2, 3)(4, 5)$$
 has  
 $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1, \pi(4) = 5, \pi(5) = 4$ 

Sym6 Six weeks of fun! Basic definitions Cycle notation

### Cycle notation is ambiguous but nice

- Many cycle notations represent the same permutation
- (1,2,3) = (2,3,1) = (3,1,2) and (1,2)(3,4) = (3,4)(1,2)
- Sometimes trivial cycles like "(5)" may be added to the notation to emphasize that  $5\in\Omega$  is a fixed point.
- The **order** of a permutation is the least positive integer *n* such that the composition of the permutation with itself *n* times is the identity
- Computing orders of permutations is easy from cycle notation. The order of a cycle is the number of moved points (its cycle length), and the order of a permutation is the least common multiple of its cycle lengths.

Sym6 Six weeks of fun! Conjugacy and centralizers Cycle type

# Cycle type and conjugacy

- The order of a permutation only depends on cycle lengths
- Cycle behave nicely under conjugation: For  $\tau = (a_1, a_2, ..., a_n)$ and a permutation  $\pi$ , applying  $\pi^{-1}$  first, then  $\tau$ , then  $\pi$  is the same as applying the cycle  $(\pi(a_1), \pi(a_2), ..., \pi(a_n))$
- The **cycle type** of a permutation is the sequence whose *n*th term is the number of cycles of length *n* in any cycle notation for the permutation
- The cycle type is well-defined, and does not change under conjugation
- In fact, two permutations are conjugate if and only if they have the same cycle type
- To convert (1,2,3)(4,5) to (1,3,4)(2,5) conjugate by  $\pi$  such that  $\pi(1) = 1, \pi(2) = 3, \pi(3) = 4, \pi(4) = 2, \pi(5) = 5$

Sym6 Six weeks of fun! Conjugacy and centralizers Centralizers

### Centralizers

- The **centralizer** of a permutation is the set of all permutations which commute with it
- The centralizer of a cycle τ is the set of all elements τ<sup>n</sup>π where n is an integer and π does not move any moved point of τ

$$C_{Sym(n)}(\tau) = \langle \tau \rangle \times Sym(Fix(\tau))$$

The centralizer of product of cycles π = τ<sub>1</sub> · · · τ<sub>n</sub> of equal lengths contains the intersections of the centralizers of the τ<sub>i</sub>. The quotient is isomorphic to Sym(n) and a quotient element σ will act by taking τ<sub>i</sub> to τ<sub>σ(i)</sub>

$$C_G(\tau_1\cdots\tau_n) = (\langle \tau_1 \rangle \times \cdots \times \langle \tau_n \rangle) \ltimes Sym(n) \times Sym(Fix(\tau))$$

• The centralizer of a permutation is the intersection of the centralizers of the products of all of its cycles of equal lengths

### Counting conjugacy classes

- Given a cycle type, there are *n*! ways to fill it in with numbers
- But each element of the centralizer gives a different way of filling in the numbers and getting the same permutation (but different cycle notation)
- So to find the number of distinct permutations with the same cycle type we divide *n*! by the size of the centralizer
- If the cycle type has  $a_n$  cycles of length n, then the centralizer has size:

$$\prod_{n=1}^{\infty} \left( n^{a_n}(a_n!) \right)$$

• Counting the number of cycle types is harder, and is the same as counting partitions

#### Pretty picture interlude

• Filling in cycle notation is like filling a Ferrer diagram

(?,?,?)(?,?,?)(?,?)(?)

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#### Pretty picture interlude

• Filling in cycle notation is like filling a Ferrer diagram

(1,2,3)(4,5,6)(7,8)(9)



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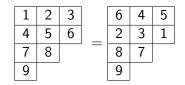
$$(1,2,3)(4,5,6)(7,8)(9) = (6,4,5)(2,3,1)(8,7)(9)$$

1	2	3
4	5	6
7	8	
9		

#### Pretty picture interlude

• Filling in cycle notation is like filling a Ferrer diagram

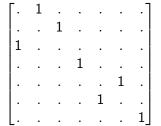
$$(1,2,3)(4,5,6)(7,8)(9) = (6,4,5)(2,3,1)(8,7)(9)$$



Sym6 Six weeks of fun! Alternate represenations Matrix representation

### Permutations as matrices

- Permutations can act on vectors as well as points
- Given a basis  $\{b_i : i \in \Omega\}$ , natural action of  $\pi$  takes  $\sum \alpha_i b_i$  to  $\sum \alpha_i b_{\pi(i)}$
- With respect to this basis, matrix has all rows and columns with exactly one 1 and the rest zeros.
- Example: (1,2,3)(5,6) acts on a seven dimensional space as the matrix



Sym6 Six weeks of fun! Alternate represenations Matrix representation

### Permutation representations

- A permutation representation is a homomorphism from a group into Sym(n)
- A permutation representation may be thought of as a matrix representation; use permutation matrices instead of permutations
- One can then apply representation theory to break the representation into smaller pieces, and understand the group
- As a silly aside, note that  $\pi(i) = i$  if and only if the (i, i)th entry of the matrix version of  $\pi$  is 1, otherwise it is 0
- So the number of fixed points is equal to the sum of the diagonal entries of the matrix; this sum is called the **trace**

Sym6 Six weeks of fun! Alternate represenations Permutation representation

### Earlier example of permutation representation

- When we looked at Sym(9) acting on the cycle notations for the permutation (1, 2, 3)(4, 5, 6)(7, 8)(9), we noticed the centralizer were those that acted trivially on the *permutation*, though not on the *notation*
- The rest of the group Sym(9) took the permutation to a different permutation of the same cycle type, so we have a permutation representation of Sym(9) acting on the conjugacy class of (1,2,3)(4,5,6)(7,8)(9)
- The subgroup that acted trivially on the one point "(1,2,3)(4,5,6)(7,8)(9)" was interesting
- It is nice that no element of Sym(9) except the identity acts trivially on all the elements of the conjugacy class

Sym6 Six weeks of fun! Alternate represenations Orbit stabilizer

# Counting elements

- The set of all points π(x) for a fixed x and varying π ∈ G is called the **orbit** of x under G
- If π(x) = σ(x), then σ<sup>-1</sup>(π(x)) = x and the composition of σ<sup>-1</sup> and π has x has a fixed point
- The subgroup of all permutations that have x ∈ Ω as a fixed point is called the stabilizer of x
- If two elements take x to the same place, then their quotient lies in in the stabilizer, and conversely:
- If two permutations lie in the same coset of the stabilizer, then they take the point to the same place
- The number of points in the orbit is equal to the size of the group divided by the size of the stabilizer of a point
- The stabilizer of  $\pi(x)$  is the conjugate by  $\pi$  of the stabilizer of x

Sym6 Six weeks of fun! Alternate represenations Transitive groups

# Transitive groups

- If  $\Omega$  consists of only one orbit, then the action is said to be  $\ensuremath{\textit{transitive}}$
- $\bullet\,$  There is then a bijection between  $\Omega$  and the cosets of the stabilizer of a point
- Choosing a different point chooses a different conjugate of the stabilizer
- The elements which act trivially on all points are those in the intersection of all conjugates of the stabilizer
- $\bullet\,$  Given a group and a conjugacy class of subgroups, one can just take  $\Omega$  to be the cosets of one specific subgroup in the conjugacy class
- The idea of a group action, and the idea of a conjugacy class of subgroups of a group are equivalent

### Blocks and subgroups above the stabilizer

- If we have a chain of subgroups H ≤ K ≤ G, then the orbits of K are permuted by the elements of G, giving a new group action
- The stabilizer of this action is K, its orbits are called **blocks**
- This is what happened in the centralizer of a product of cycles of the same length
- *H* is the intersection of the centralizers of the cycles, *K* is centralizer of their product, *K* acts by permuting the cycles

Sym6 Six weeks of fun! Alternate represenations Young tabloid

### Final example: Young tabloid

- We can define a new equivalence relation on the diagrams: two diagrams are row equivalent if there is a permutation that takes one to the other while fixing the rows setwise
- The stabilizer of a diagram is easy to describe:
- If the diagram is filled in as

• then the stabilizer is

 $\textit{Sym}(\{1,2,3\}) \times \textit{Sym}(\{4,5,6\}) \times \textit{Sym}(\{7,8\}) \times \textit{Sym}(\{9\})$ 

# Conclusion

- Permutations are more easily understood in cycle notation: can read off
  - Order
  - ② Conjugacy class
  - ③ Centralizer
- Permutations can act on vectors and the trace is an important and natural invariant
- Other groups can act as permutations
- Through the magic of stabilizers, these actions are just actions on cosets
- Blocks encode the idea of "subgroup"
- Understanding permutations means understanding subgroups