Part I

Sym(6): Six weeks of fun!

Goals and expectations

- My goal is to make some portion of the literature in algebra and combinatorics more accessible, especially the excellent book by B. Sagan
- This is the “movie version” of the book, where important themes are made clear, but details are freely omitted
- During the first three weeks participants are expected to come with an open mind and to read through the first two chapters.
- During the last three weeks, some volunteers will be expected to be bold and explain to us some facet of Sagan’s book

Overview of Course

- In the first three weeks we will setup an algebraic and combinatorial scaffolding so that we can work on the high points of the book safely.
- In the last three weeks we will look at the extremely interesting interaction of algebra and combinatorics present in the ordinary representation theory of the symmetric groups.
- First three weeks:
  1. Permutations and Tableaux
  2. Character theory and Robinson-Schensted
  3. Representation theory and the Branching Rule

Outline of the reading for Tuesday

- (1.1) Cycle notation, Cycle types, and Partitions
- Conjugacy in Sym(n) and Centralizers
- (1.2) Permutation matrices and representations
- Fixed point formula
- (1.3) Coset representatives
- (1.12) Transversals
- (1.6) Permutation action on Young tabloid
- Exercises: Ch1: #1,2,3,5c,11

Outline of the reading for Thursday

- (1.6) Tabloids
- (2.1) Tableaux etc.
- Theorem 2.1.12 for permutations, not modules
- (2.3) Row and column stabilizers
- Exercises: Ch2: #1,2,8,10

Part II

Week 1A: Permutations

Permutations

- A **permutation** on a set Ω is a one-to-one, onto function from Ω to Ω
- Two permutations on Ω can be composed as functions to yield another permutation
- The set of all such functions is called the **symmetric group** on Ω and denoted $Sym(\Omega)$
- When $\Omega = \{1, 2, \ldots, n\}$ we write $Sym(\Omega) = Sym(n) = S_n$
- $\#Sym(n) = n!$
- If $\pi(x) = x$, then x is a **fixed point**, otherwise it is a **moved point**.

Cycle notation

- A **cycle** $\pi = (a_1, a_2, \ldots, a_n)$ is a special type of permutation
- $\pi(x) = \begin{cases} a_{i+1} & \text{if } x = a_i, 1 \leq i < n \\ a_1 & \text{if } x = a_n \\ x & \text{otherwise} \end{cases}$
- Two cycles are **disjoint** if no point of Ω is moved by both
- **Cycle notation** for a permutation π is a formal product of disjoint cycles whose actual product is π
- Example: $\pi = (1, 2, 3)(4, 5)$ has
  $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1, \pi(4) = 5, \pi(5) = 4$

Cycle notation is ambiguous but nice

- Many cycle notations represent the same permutation
- $(1,2,3) = (2,3,1) = (3,1,2)$ and $(1,2)(3,4) = (3,4)(1,2)$
- Sometimes trivial cycles like "(5)" may be added to the notation to emphasize that $5 \in \Omega$ is a fixed point.
- The **order** of a permutation is the least positive integer n such that the composition of the permutation with itself n times is the identity
- Computing orders of permutations is easy from cycle notation. The order of a cycle is the number of moved points (its **cycle length**), and the order of a permutation is the least common multiple of its cycle lengths.

Cycle type and conjugacy

- The order of a permutation only depends on cycle lengths
- Cycle behave nicely under conjugation: For $\tau = (a_1, a_2, \ldots, a_n)$ and a permutation π, applying π^{-1} first, then τ, then π is the same as applying the cycle $(\pi(a_1), \pi(a_2), \ldots, \pi(a_n))$
- The **cycle type** of a permutation is the sequence whose nth term is the number of cycles of length n in any cycle notation for the permutation
- The cycle type is well-defined, and does not change under conjugation
- In fact, two permutations are conjugate if and only if they have the same cycle type
- To convert $(1, 2, 3)(4, 5)$ to $(1, 3, 4)(2, 5)$ conjugate by π such that $\pi(1) = 1, \pi(2) = 3, \pi(3) = 4, \pi(4) = 2, \pi(5) = 5$

Centralizers

- The **centralizer** of a permutation is the set of all permutations which commute with it
- The centralizer of a cycle τ is the set of all elements $\tau^n\pi$ where n is an integer and π does not move any moved point of τ

$$C_{Sym(n)}(\tau) = \langle\tau\rangle \times Sym(Fix(\tau))$$

- The centralizer of product of cycles $\pi = \tau_1 \cdots \tau_n$ of equal lengths contains the intersections of the centralizers of the τ_i. The quotient is isomorphic to $Sym(n)$ and a quotient element σ will act by taking τ_i to $\tau_{\sigma(i)}$

$$C_G(\tau_1 \cdots \tau_n) = (\langle\tau_1\rangle \times \cdots \times \langle\tau_n\rangle) \ltimes Sym(n) \times Sym(Fix(\tau))$$

- The centralizer of a permutation is the intersection of the centralizers of the products of all of its cycles of equal lengths

Counting conjugacy classes

- Given a cycle type, there are $n!$ ways to fill it in with numbers
- But each element of the centralizer gives a different way of filling in the numbers and getting the same permutation (but different cycle notation)
- So to find the number of distinct permutations with the same cycle type we divide $n!$ by the size of the centralizer
- If the cycle type has a_n cycles of length n, then the centralizer has size:

$$\prod_{n=1}^{\infty} (n^{a_n}(a_n!))$$

- Counting the number of cycle types is harder, and is the same as counting partitions

Pretty picture interlude

- Filling in cycle notation is like filling a Ferrer diagram

(?,?,?)(?,?,?)(?,?)(?)

Pretty picture interlude

- Filling in cycle notation is like filling a Ferrer diagram

$$(?,?,?)(?,?,?)(?,?)(?)$$

?	?	?
?	?	?
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Pretty picture interlude

- Filling in cycle notation is like filling a Ferrer diagram

(1,2,3)(4,5,6)(7,8)(9)

?	?	?
?	?	?
?	?	
?		

Pretty picture interlude

- Filling in cycle notation is like filling a Ferrer diagram

(1,2,3)(4,5,6)(7,8)(9)

1	2	3
4	5	6
7	8	
9		

Pretty picture interlude

- Filling in cycle notation is like filling a Ferrer diagram

$$(1,2,3)(4,5,6)(7,8)(9) = (6,4,5)(2,3,1)(8,7)(9)$$

1	2	3
4	5	6
7	8	
9		

Pretty picture interlude

- Filling in cycle notation is like filling a Ferrer diagram

$$(1,2,3)(4,5,6)(7,8)(9) = (6,4,5)(2,3,1)(8,7)(9)$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \cline{1-2} 9 & & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|}\hline 6 & 4 & 5 \\ \hline 2 & 3 & 1 \\ \hline 8 & 7 & \\ \cline{1-2} 9 & & \\ \cline{1-1} \end{array}$$

Permutations as matrices

- Permutations can act on vectors as well as points
- Given a basis $\{b_i : i \in \Omega\}$, natural action of π takes $\sum \alpha_i b_i$ to $\sum \alpha_i b_{\pi(i)}$
- With respect to this basis, matrix has all rows and columns with exactly one 1 and the rest zeros.
- Example: $(1, 2, 3)(5, 6)$ acts on a seven dimensional space as the matrix

$$\begin{bmatrix} . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ 1 & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 \end{bmatrix}$$

Permutation representations

- A permutation representation is a homomorphism from a group into Sym(n)
- A permutation representation may be thought of as a matrix represetation; use permutation matrices instead of permutations
- One can then apply representation theory to break the representation into smaller pieces, and understand the group
- As a silly aside, note that $\pi(i) = i$ if and only if the (i, i)th entry of the matrix version of π is 1, otherwise it is 0
- So the number of fixed points is equal to the sum of the diagonal entries of the matrix; this sum is called the **trace**

Earlier example of permutation representation

- When we looked at Sym(9) acting on the cycle notations for the permutation $(1,2,3)(4,5,6)(7,8)(9)$, we noticed the centralizer were those that acted trivially on the *permutation*, though not on the *notation*
- The rest of the group Sym(9) took the permutation to a different permutation of the same cycle type, so we have a permutation representation of Sym(9) acting on the conjugacy class of $(1,2,3)(4,5,6)(7,8)(9)$
- The subgroup that acted trivially on the one point "$(1,2,3)(4,5,6)(7,8)(9)$" was interesting
- It is nice that no element of Sym(9) except the identity acts trivially on all the elements of the conjugacy class

Counting elements

- The set of all points $\pi(x)$ for a fixed x and varying $\pi \in G$ is called the **orbit** of x under G
- If $\pi(x) = \sigma(x)$, then $\sigma^{-1}(\pi(x)) = x$ and the composition of σ^{-1} and π has x has a fixed point
- The subgroup of all permutations that have $x \in \Omega$ as a fixed point is called the **stabilizer** of x
- If two elements take x to the same place, then their quotient lies in in the stabilizer, and conversely:
- If two permutations lie in the same coset of the stabilizer, then they take the point to the same place
- The number of points in the orbit is equal to the size of the group divided by the size of the stabilizer of a point
- The stabilizer of $\pi(x)$ is the conjugate by π of the stabilizer of x

Transitive groups

- If Ω consists of only one orbit, then the action is said to be **transitive**
- There is then a bijection between Ω and the cosets of the stabilizer of a point
- Choosing a different point chooses a different conjugate of the stabilizer
- The elements which act trivially on all points are those in the intersection of all conjugates of the stabilizer
- Given a group and a conjugacy class of subgroups, one can just take Ω to be the cosets of one specific subgroup in the conjugacy class
- The idea of a group action, and the idea of a conjugacy class of subgroups of a group are equivalent

Blocks and subgroups above the stabilizer

- If we have a chain of subgroups $H \leq K \leq G$, then the orbits of K are permuted by the elements of G, giving a new group action
- The stabilizer of this action is K, its orbits are called **blocks**
- This is what happened in the centralizer of a product of cycles of the same length
- H is the intersection of the centralizers of the cycles, K is centralizer of their product, K acts by permuting the cycles

Final example: Young tabloid

- We can define a new equivalence relation on the diagrams: two diagrams are row equivalent if there is a permutation that takes one to the other while fixing the rows setwise
- The stabilizer of a diagram is easy to describe:
- If the diagram is filled in as

1	2	3
4	5	6
7	8	
9		

- then the stabilizer is

$$Sym(\{1,2,3\}) \times Sym(\{4,5,6\}) \times Sym(\{7,8\}) \times Sym(\{9\})$$

Conclusion

- Permutations are more easily understood in cycle notation: can read off
  1. Order
  2. Conjugacy class
  3. Centralizer
- Permutations can act on vectors and the trace is an important and natural invariant
- Other groups can act as permutations
- Through the magic of stabilizers, these actions are just actions on cosets
- Blocks encode the idea of “subgroup”
- Understanding permutations means understanding subgroups

THE END