Part III

Week 2A: Character Theory

Be more concerned
with your character
than your representation

- UCLA basketball coach John Wooden's
take on modern group theory

Character theory in a nutshell

- Character theory uses simple and natural numerical invariants to describe groups
- Most group theoretic concepts are influenced by character theory
- Character tables are often easier to work with than groups, and almost always preferable to irreducible representations

Character theory is useful

- Character theory forms a large part of natural proofs of the following nice results:

1. Groups of order $p^a q^b$ are solvable
2. If a 2-group has exactly $4k + 1$ elements of order 2, then it is cyclic, dihedral, quaternion, or semidihedral.
3. For a group H, there are only finitely many simple groups G containing an involution t with $C_G(t) = H$. For instance, if H is order 8 dihedral, then $G = PSL(2, 7)$ or $G = PSL(2, 9)$.
4. The simple groups with elementary abelian, self-centralizing Sylow 2-subgroup are precisely $SL(2, 2^k)$
5. No nonabelian simple group has a cyclic Sylow 2-subgroup, nor a quaternion Sylow 2-subgroup

Character theory reflects group theory

- A group G has the form $P \ltimes K$ for K a normal abelian p'-group, P a p-group if and only if every character has degree a power of p. If every character has degree merely divisible by p, then K need not be abelian.

- A group has the form $K \ltimes P$ for K a p'-subgroup and P a normal p-subgroup if and only if the p-part of every irreducible character is a character.

- The character table determines:

  1. The lattice of normal subgroups, including their size and the property of having Abelian quotient

  2. Whether the group is solvable (of what derived length), whether the group is nilpotent (of what class), and the commutator length

Character theory is compatible with combinatorics

- Many of the earliest and the most important applications of character theory are to counting
- Character values are (cyclotomic) integers
- Characters are nonnegative sums of irreducible characters, and decomposing known charcters into unknown irreducibles is attacked through various combinatorial methods
- For the symmetric group, character values are plain old integers that can be read off as coefficients of classical polynomials
- Characters are studied using combinatorial objects such as Brauer trees (and graphs)

Ok, I'm sold! What did I buy?

- Characters are defined from representations

- A **representation** of a finite group G is a homomorphism from G into the group of $n \times n$ complex matrices, $GL(n, \mathbb{C})$

- A **character** of a finite group is a function χ from G to $\mathbb{C}$ such that there is some representation X of G with $\chi(g)$ equal to the trace of $X(g)$

- The trace is invariant under change of basis (it is the sum of the eigenvalues) and easy to calculate (it is the sum of the diagonal entries)

Basic properties

- The definition of $\chi(g)$ as trace gives a number of properties:

1. $\chi(g)$ is a sum of $|g|$th roots of unity
2. $|\chi(g)| \leq \chi(1)$, which is the dimension of the representation $(1 + 1 + \ldots + 1)$, called the **degree** of χ
3. $\chi(g) = \chi(1)$ if and only if $X(g) = X(1)$
4. $|\chi(g)| = \chi(1)$ if and only if $X(g) = \chi(g)/\chi(1) \cdot X(1)$ is a scalar matrix
5. $\overline{\chi(g)} = \chi(g^{-1})$

More definitions and properties

- A representation is **irreducible** if the *G*-orbit of every nonzero vector is a spanning set
- The **inner product** of two characters χ, ψ is

$$[\chi, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1})$$

- There are finitely many irreducible characters
- They form an orthonormal basis of the vector space of all complex functions constant on conjugacy classes of *G*
- $\psi = \sum_{\chi \in Irr(G)} m_\chi \chi$ if and only if $m_\chi = [\psi, \chi]$
- A character ψ is irreducible if and only if $[\psi, \psi] = 1$

More properties

- Two elements $g, h \in G$ are conjugate in G if and only if $\chi(g) = \chi(h)$ for every irreducible character of G
- Two representations are isomorphic as G-modules if and only if their characters are equal
- If χ is irreducible, then $\chi(1)$ divides $|G|$, in fact for any abelian subnormal subgroup A, $\chi(1)|[G : A]$
- The number irreducible characters is equal to the number of conjugacy classes
- $|G| = \sum_{\chi \in Irr(G)} \chi(1)^2$

New characters from old

- **Induction** expands a charcter from a subgroup to the whole group, much like a permutation matrix expands "1" from a subgroup to the whole group via the action of the group on cosets

- **Inflation** expands a character from a quotient group to the whole group in a very nice way:

$$G \twoheadrightarrow G/N \to GL(n, \mathbb{C})$$

- **Restriction** takes a character from the whole group to a subgroup

- **Tensor product** takes a character from two groups and forms one for their direct product

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All characters come from old permutation characters

- If G is a permutation group with permutation character χ, then all characters of G are constituents of tensor powers of χ
- But how do we figure out how to break down a character?
- For the symmetric groups, some permutation characters come in a nice order, and the only new irreducible characters occur exactly one at a time, so

$$\text{New} = \text{Perm} - \sum_{\chi \text{ known}} [\chi, \text{Perm}]\chi$$

- The character of a permutation character is the number of fixed points. If K is a point stabilizer, then we want to count how often $ghK = hK$, that is how often $h^{-1}gh \in K$:

$$\chi(g) = \#\{h \in G : h^{-1}gh \in K\}/\#K$$

Some examples

- Sym(1) has only one conjugacy class, so only one irreducible character. It is the character $\chi(g) = 1$
- Sym(2) has two conjugacy classes.
  1. One irreducible character is the inflation of the character of Sym(1) = Sym(2) / Sym(2), $\chi(g) = 1$
  2. The induced character is just the permutation character of Sym(2) acting on the cosets of Sym(1), so here the values are the number of fixed points, $\psi(1) = 2$ and $\psi(1,2) = 0$, but $[\psi, \psi] = \frac{1}{2}(2^2 + 0) = 2$ so it is not irreducible
  3. Luckily $[\psi, \chi] = 1$ so $\theta = \psi - \chi$ is an irreducible character
- Sym(3) follows a similar path, three conjugacy classes. First irreducible character inflated from Sym(1), second found by fixing the induced character from Sym(2), the third found by fixing the induced character from Sym(1)

Perm. and irred. characters of Sym(3)

Sym(3)	1 $(1,1,1)$	3 $(2,1)$	2 (3)
$P_{(3)}$	1	1	1
$P_{(2,1)}$	3	1	0
$P_{(1,1,1)}$	6	0	0

Sym(3)	1 $(1,1,1)$	3 $(2,1)$	2 (3)
$S_{(3)}$	1	1	1
$S_{(2,1)}$	2	0	-1
$S_{(1,1,1)}$	1	-1	1

$S_{(3)} = P_{(3)}$ since $\langle P_{(3)}, P_{(3)} \rangle = 1$

$S_{(2,1)} = P_{(2,1)} - S_{(3)}$ since $\left\langle P_{(2,1)}, S_{(3)} \right\rangle = 1$

$S_{(1,1,1)} = P_{(1,1,1)} - S_{(3)} - 2S_{(2,1)}$ since $\left\langle P_{(1,1,1)}, S_{(3)} \right\rangle = 1$

and $\left\langle P_{(1,1,1)}, S_{(2,1)} \right\rangle = 2$

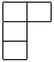
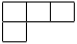
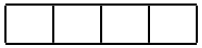
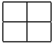
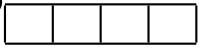
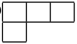
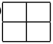
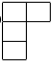
More examples

- Sym(4) has five conjugacy classes, so five irreducible characters

- Begin with the permutation characters induced from Young subgroups

- Note that $gH = kgH$ if and only if $k^g \in H$, so to calculate a permutation character, we only need to count how many elements conjugate k into H and divide $|H|$

Permutation characters of Sym(4)

Sym(4)	1 	6 	8 	6 	3 
P	1	1	1	1	1
P	4	2	1	0	0
P	6	2	0	0	2
P	12	2	0	0	0
P	24	0	0	0	0

Scalar products for permutation characters

- $\left\langle P_{\yng(4)}, P_{\yng(4)} \right\rangle =$
  $(1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 6 + 1 \cdot 1 \cdot 8 + 1 \cdot 1 \cdot 6 + 1 \cdot 1 \cdot 3)/24 = 1$

- $\left\langle P_{\yng(3,1)}, P_{\yng(4)} \right\rangle =$
  $(4 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 6 + 1 \cdot 1 \cdot 8 + 0 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 3)/24 = 1$

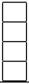
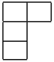
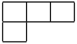
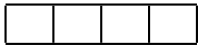
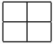
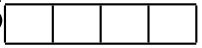
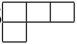
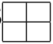
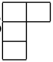
- $\left\langle P_{\yng(3,1)}, P_{\yng(3,1)} \right\rangle = 2$, so $P_{\yng(3,1)} - P_{\yng(4)}$ is irreducible

- Define $S_{\yng(4)} = P_{\yng(4)}$ to be the first irreducible character we found

- $S_{\yng(3,1)} = P_{\yng(3,1)} - P_{\yng(4)}$ to be the second

- Continue in this fashion to subtract off the known *S* characters from the *P* characters

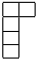
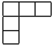
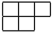
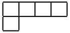
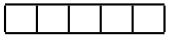
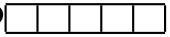
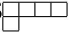
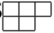
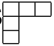
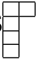
Irreducible characters of Sym(4)

	1	6	8	6	3
Sym(4)					
S	1	1	1	1	1
S	3	1	0	-1	-1
S	2	0	-1	0	2
S	3	-1	0	1	-1
S	1	-1	1	-1	1

Permutation characters of Sym(5)

Sym(5)	1	10	15	20	20	30	24
	$(1,1,1,1,1)$	$(2,1,1,1)$	$(2,2,1)$	$(3,1,1)$	$(3,2)$	$(4,1)$	(5)
$P_{(5)}$	1	1	1	1	1	1	1
$P_{(4,1)}$	5	3	1	2	0	1	0
$P_{(3,2)}$	10	4	2	1	1	0	0
$P_{(3,1,1)}$	20	6	0	2	0	0	0
$P_{(2,2,1)}$	30	6	2	0	0	0	0
$P_{(2,1,1,1)}$	60	6	0	0	0	0	0
$P_{(1,1,1,1,1)}$	120	0	0	0	0	0	0

Irreducible characters of Sym(5)

Sym(5)	1 	10 	15 	20 	20 	30 	24 
S 	1	1	1	1	1	1	1
S 	4	2	0	1	-1	0	-1
S 	5	1	1	-1	1	-1	0
S 	6	0	-2	0	0	0	1
S 	5	-1	1	-1	-1	1	0
S 	4	-2	0	1	1	0	-1
S 	1	-1	1	1	-1	-1	1

Conclusion

- Characters are wonderfully useful (claimed, not shown)
- Characters are easy to work with (hopefully shown)
- Characters are fun and pretty
- When one has trouble understanding what might be true at the module or matrix level, see what it says for characters

THE END